TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 356, Number 7, Pages 2889–2925
S 0002-9947(04)03558-5
Article electronically published on March 2, 2004

A SEPARABLE BROWN-DOUGLAS-FILLMORE THEOREM AND WEAK STABILITY

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ABSTRACT. We give a separable Brown-Douglas-Fillmore theorem. Let A be a separable amenable C^*-algebra which satisfies the approximate UCT, B be a unital separable amenable purely infinite simple C^*-algebra and $h_1, h_2 : A \to B$ be two monomorphisms. We show that h_1 and h_2 are approximately unitarily equivalent if and only if $[h_1] = [h_2]$ in $KL(A, B)$. We prove that, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: for any amenable purely infinite simple C^*-algebra B and for any contractive positive linear map $L : A \to B$ such that

$$\|L(ab) - L(a)L(b)\| < \delta \quad \text{and} \quad \|L(a)\| \geq (1/2)\|a\|$$

for all $a \in \mathcal{G}$, there exists a homomorphism $h : A \to B$ such that

$$\|h(a) - L(a)\| < \varepsilon \quad \text{for all } a \in \mathcal{F}$$

provided, in addition, that $K_i(A)$ are finitely generated. We also show that every separable amenable simple C^*-algebra A with finitely generated K-theory which is in the so-called bootstrap class is weakly stable with respect to the class of amenable purely infinite simple C^*-algebras. As an application, related to perturbations in the rotation C^*-algebras studied by U. Haagerup and M. Rørdam, we show that for any irrational number θ and any $\varepsilon > 0$ there is $\delta > 0$ such that in any unital amenable purely infinite simple C^*-algebra B if

$$\|uv - e^{i\theta\pi}vu\| < \delta$$

for a pair of unitaries, then there exists a pair of unitaries u_1 and v_1 in B such that

$$u_1v_1 = e^{i\theta\pi}v_1u_1, \quad \|u_1 - u\| < \varepsilon \quad \text{and} \quad \|v_1 - v\| < \varepsilon.$$

1. INTRODUCTION

The Brown-Douglas-Fillmore theorem of classification of essentially normal operators ([BDF]) is a milestone in both operator theory and operator algebras. One may regard the BDF-theory as a classification of homomorphisms from $C(X)$ into the Calkin algebra. As in topology, where the study of continuous maps from one space to another is fundamentally important, the study of homomorphisms from one C^*-algebra to another is also very important. The Calkin algebra is a special non-separable purely infinite simple C^*-algebra. It turns out that it is both interesting and important to study homomorphisms (monomorphisms) from a separable

Received by the editors September 18, 2002 and, in revised form, April 29, 2003.
2000 *Mathematics Subject Classification.* Primary 46L05, 46L80.
Key words and phrases. Weakly semiprojective C^*-algebras, purely infinite simple C^*-algebras.
This research was partially supported by NSF grant DMS 0097903.

C^*-algebra to a (separable) purely infinite simple C^*-algebra. In fact, the classification theorem of Kirchberg and Phillips for separable amenable purely infinite simple C^*-algebras can be regarded as a result of classification of homomorphisms from one separable amenable purely infinite simple C^*-algebra to another.

In this paper, we will study maps from separable amenable C^*-algebras to amenable purely infinite simple C^*-algebras. Combining a uniqueness theorem of the author with the methods of N. C. Phillips in [P3] and results in [KP], we prove the following classification of injective homomorphisms: Let A be a unital separable amenable C^*-algebra which satisfies the approximate UCT (something weaker than UCT) and let B be a unital separable amenable purely infinite simple C^*-algebra. Suppose that h_1, $h_2 : A \to B$ are two unital monomorphisms. Then there exists a sequence of unitaries $u_n \in B$ such that

$$\lim_{n\to\infty} \|\mathrm{ad}\, u_n \circ h_1(a) - h_2(a)\| = 0$$

for all $a \in A$ if and only if $[h_1] = [h_2]$ in $KL(A, B)$. Moreover, we also show that every element in $KL(A, B)$ can be represented by a homomorphism. When both $K_0(A)$ and $K_1(A)$ are torsion free, the above says that two monomorphisms are approximately unitarily equivalent if they induce the same map on $K_0(A)$ and $K_1(A)$. This may be viewed as a separable and non-commutative version of the classical BDF-theorem.

It was shown in [Ln1] that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any pair of unitaries u and v in a purely infinite simple C^*-algebra B satisfying

$$\|uv - vu\| < \delta$$

there exists a commuting pair of unitaries w and z in B such that

$$\|u - w\| < \varepsilon \quad \text{and} \quad \|v - z\| < \varepsilon.$$

Related problems in perturbation of rotation C^*-algebras and the Heisenberg commutation relation were also studied in [HR] by U. Haagerup and M. Rørdam. A closely-related question was raised. Let θ be an irrational number. Suppose that there is a pair of unitaries u and v in B such that $uv \neq e^{i\theta\pi} vu$ but $\|uv - e^{i\theta\pi} vu\|$ is small. Does there exist a pair of unitaries w and z in B which has the rotation property that $wz = e^{i\theta\pi} zw$ and $\|w - u\|$ and $\|z - v\|$ are small? We will answer this question affirmatively.

We study a much more general problem. Let $\mathbf{B}$ be the class of amenable purely infinite simple C^*-algebras and let A be a unital separable amenable C^*-algebra. We study the following problem: Which separable amenable C^*-algebras are weakly stable (see 2.2) with respect to $\mathbf{B}$? In other words, we want to know when A satisfies the following: For any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, is there $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ such that for any contractive positive linear map $L : A \to B$ for any $B \in \mathbf{B}$ which satisfies

$$\|L(ab) - L(a)L(b)\| < \delta \ \text{ for all } \ a \in \mathcal{G}$$

there exists a homomorphism $h : A \to B$ such that

$$\|h(a) - L(a)\| < \varepsilon \ \text{ for all } \ a \in \mathcal{F}?$$

We find that the last question is closely related to the classification of homomorphisms from A to B. To avoid complication caused by quotients of A and to simplify the terminology, one may assume L to be "approximately injective". For example, we may further assume that $\|L(a)\| \geq (1/2)\|a\|$ for all $a \in \mathcal{G}$. With this additional restriction, we prove that any unital separable amenable C^*-algebra A which satisfy the approximate UCT and have finitely generated $K_i(A)$ $(i = 0, 1)$ are (approximately injective) weakly stable with respect to the class **B**. To remove this additional "approximately injective" condition, we show that any unital separable amenable simple C^*-algebras which satisfy the AUCT and have finitely generated $K_i(A)$ $(i = 0, 1)$ are in fact weakly stable with respect to **B**. Since irrational rotation C^*-algebras are simple, this result gives an affirmative answer to the question about perturbation of rotation C^*-algebras mentioned above.

(Relative) weakly stable C^*-algebras and (relative) weakly semiprojective C^*-algebras have been studied before. A nice treatment can be found in T. Loring's book [Lo3]. However, very few finite separable amenable C^*-algebras are known to be (relative) weakly stable or weakly semiprojective. In this sense, the results obtained in this paper (7.5, 7.6, 7.7 and 7.8) are somewhat unexpected.

The condition that $K_i(A)$ are finitely generated is not necessary regarding the problem about relative weakly stability of the unital separable amenable C^*-algebras. For example, for most cases, the results about weakly stability hold when $K_i(A)$ are direct sums of finitely generated abelian groups. However, it is not possible to proceed further from here. To be any form of weakly stable (or weakly semiprojective), $K_i(A)$ has to be a direct sum of finitely generated abelian groups. In particular, we show that no UHF-algebras are weakly stable with respect to **B**.

The paper is organized as follows. Section 2 lists some of the notation and conventions that are used in the paper. Section 3 is devoted to the study of maps to $\mathcal{O}_2$. It is shown that all separable amenable C^*-algebras are "approximately injective" weakly stable with respect to $\mathcal{O}_2$ and all separable amenable simple C^*-algebras are weakly semiprojective with respect to $\mathcal{O}_2$. We give some preliminary results about approximately multiplicative sequences of contractive completely positive linear maps in section 4. In section 5 we give a sequential version of Phillips's results on asymptotic morphisms. In section 6, we give the classification of monomorphisms from a separable amenable C^*-algebras to separable amenable purely infinite simple C^*-algebras. We study the weak stability of separable amenable C^*-algebras in section 7. We also present one of the main theorems on weak stability there. In the last section, we show that for every (pf) weakly stable C^*-algebra A, $K_*(A)$ must be a countable direct sum of finitely generated abelian groups.

2. Preliminaries

We will use the following conventions:

Let A and B be C^*-algebras and $\phi, \psi : A \to B$ be two maps. Let $\varepsilon > 0$ and $\mathcal{F} \subset A$.

(1) We write

$$\phi \approx_\varepsilon \psi \text{ on } \mathcal{F}$$

if

$$\|\phi(a) - \psi(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}.$$

(2) We write

$$\phi \overset{u}{\sim}_\varepsilon \psi \text{ on } \mathcal{F}$$

if there is a unitary U in B (or in $\tilde{B}$ if B is not unital) such that

$$\|\mathrm{ad}\, U \circ \phi(a) - \psi(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}.$$

(3) We write

$$\phi \stackrel{u}{\sim} \psi$$

if there exists a sequence of unitaries $\{u_n\}$ in B (or in $\tilde{B}$ if B is not unital) such that

$$\lim_{n\to\infty} \|\mathrm{ad}\, u_n \circ \phi(a) - \psi(a)\| = 0$$

for all $a \in A$.

(4) Let $\{B_n\}$ be a sequence of C^*-algebras. We will use the following notation: $c_0(\{B_n\}) = \bigoplus_{n=1}^{\infty} B_n$, $l^\infty(\{B_n\}) = \prod_{n=1}^{\infty} B_n$ and $q_\infty(\{B_n\}) = l^\infty(\{B_n\})/c_0(\{B_n\})$. For the case in which $B_n = B$ for all n, we will use $c_0(B)$ for $c_0(\{B_n\})$, $l^\infty(B)$ for $l^\infty(\{B_n\})$ and $q_\infty(B)$ for $q_\infty(\{B_n\})$, respectively.

(5) A linear map $L : A \to B$ is said to be *full* if the ideal generated by $L(a)$ is B for any non-zero $a \in A$.

(6) Let A be a C^*-algebra. We denote by $d_n : A \to M_n(A)$ the map defined by $d_n(a) = \mathrm{diag}(a, a, ..., a)$, where $a \in A$ is repeated n times on the diagonal.

(7) Let $L : A \to B$ be a linear map, let $\mathcal{G}$ be a subset of A and let $\varepsilon > 0$. We say L is $\mathcal{G}$-ε-multiplicative if

$$\|L(ab) - L(a)L(b)\| < \varepsilon \quad \text{for all } a \quad \text{and} \quad b \in \mathcal{G}.$$

Definition 2.1. Let A and B be two C^*-algebras. Recall that a contractive completely positive linear map $\phi : A \to B$ is *amenable* if for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exist two contractive completely positive linear maps $L_1 : A \to M_k$ (for some integer $k > 0$) and $L_2 : M_k \to B$ such that

$$\phi \approx_\varepsilon L_2 \circ L_1 \quad \text{on } \mathcal{F}.$$

Recall that a C^*-algebra is said to be amenable if id_A is amenable.

Definition 2.2. Fix a class of C^*-algebras $\mathbf{D}$. Let A be a separable amenable C^*-algebra. We say A is *weakly stable with respect to* $\mathbf{D}$ if, for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following: for any $B \in \mathbf{D}$ and any positive linear contraction $L : A \to B$ which is $\mathcal{G}$-δ-multiplicative, there exists a homomorphism $h : A \to B$ such that

$$h \approx_\varepsilon L \quad \text{on } \mathcal{F}.$$

It should be noted that here δ and $\mathcal{G}$ depend only on ε and $\mathcal{F}$. They do not depend on B.

Definition 2.3. Fix a class of C^*-algebras $\mathbf{D}$. Let A be a separable amenable C^*-algebra. We say that A is *weakly semiprojective* with respect to $\mathbf{D}$, if for any sequence $B_n \in \mathbf{D}$ and a homomorphism $\phi : A \to q_\infty(\{B_n\})$, there exists a homomorphism $h : A \to l^\infty(\{B_n\})$ such that $\pi \circ h = \phi$, where $\pi : l^\infty(\{B_n\}) \to q_\infty(\{B_n\})$ is the quotient map.

This definition can be found in T. Loring's book [Lo3] with a slight modification.

Theorem 2.4 (cf. 19.1.3 in [Lo3])**.** *Let* $\mathbf{D}$ *be a class of* C^**-algebras and let* A *be a separable amenable* C^**-algebra. Then* A *is weakly stable with respect to* $\mathbf{D}$ *if and only if it is weakly semiprojective with respect to* $\mathbf{D}$.

Proof. Suppose that A is weakly stable with respect to $\mathbf{D}$. Let $\{B_n\}$ be a sequence of C^*-algebras in $\mathbf{D}$ and let $\phi : A \to q_\infty(\{B_n\})$ be a homomorphism. Suppose that $\{\mathcal{F}_n\}$ is an increasing sequence of finite subsets of A such that $\bigcup_n \mathcal{F}_n$ is dense in A. Let $\{\mathcal{G}_n\}$ be finite subsets of A and $\delta_n > 0$ satisfying the following: for any

$\mathcal{G}_n$-δ_n-multiplicative contractive completely positive linear map $L : A \to B$ (for any $B \in \mathbf{D}$) there exists a homomorphism $\phi : A \to B$ such that $\phi \approx_{1/n} L$ on $\mathcal{F}_n$. Since A is amenable, by [CE], there exists a contractive completely positive linear map $\Phi : A \to l^\infty(\{B_n\})$ such that $\pi \circ \Phi = \phi$, where $\pi : l^\infty(\{B_n\}) \to q_\infty(\{B_n\})$ is the quotient map. Write $\Phi = \{L_n\}$, where $L_n : A \to B_n$ is a contractive completely positive linear map. Then

$$\lim_{n\to\infty} \|L_n(ab) - L_n(a)L_n(b)\| = 0 \text{ for all } a, b \in A.$$

There is, for each k, an integer $n(k)$ such that $n(k+1) > n(k)$ and

$$\|L_n(ab) - L_n(a)L_n(b)\| < \delta_k \text{ for all } a, b \in \mathcal{G}_k \quad \text{and} \quad n(k) < n.$$

By the choice of $\mathcal{G}_n$ and δ_n, we obtain homomorphisms $h_n : A \to B_n$ such that

$$\|L_n(a) - h_n(a)\| < 1/k \text{ for } k(n) < n \le n(k+1).$$

Define $h = \{h_n\}$. Then $\pi \circ h = \phi$. Thus A is weakly semiprojective with respect to $\mathbf{D}$.

Now we assume that A is weakly semiprojective with respect to $\mathbf{D}$ and is not weakly stable. Thus, for some $\varepsilon_0 > 0$ and some finite subset $\mathcal{F}_0 \subset A$, there exists a sequence of $B_n \in \mathbf{D}$ and a sequence of $\mathcal{F}_n$-$1/2^n$-multiplicative contractive completely positive linear maps $L_n : A \to B_n$ such that

$$\lim_{n\to\infty} \|L_n(ab) - L_n(a)L_n(b)\| = 0 \text{ for all } a, b \in A$$

and

$$\inf\{\sup\{\|L_n(a) - h(a)\| : a \in \mathcal{F}_0\}\} \ge \varepsilon_0/2,$$

where the infimum is taken among all homomorphisms h from A into B_n. Let $L : A \to l^\infty(\{B_n\})$ be defined by $L = \{L_n\}$ and $\pi : l^\infty(\{B_n\}) \to q_\infty(\{B_n\})$ be the quotient map. Note that $\phi = \pi \circ L : A \to q_\infty(\{B_n\})$ is a homomorphism. Since A is weakly semiprojective with respect to $\mathbf{D}$, there is a homomorphism $h : A \to l^\infty(\{B_n\})$ such that $\pi \circ h = \phi$. Write $h = \{h_n\}$, where $h_n : A \to B_n$ is a homomorphism. Then

$$\lim_{n\to\infty} \|L_n(a) - h_n(a)\| = 0$$

for all $a \in A$. This contradicts the assumption that

$$\inf\{\sup\{\|L_n(a) - h(a)\| : a \in \mathcal{F}_0\}\} \ge \varepsilon_0/2.$$

Therefore A is weakly stable with respect to $\mathbf{D}$. □

In what follows, we will not distinguish weakly stable from weakly semiprojective (with respect to $\mathbf{D}$).

Definition 2.5. Fix a class of C^*-algebras $\mathbf{D}$. Let A be a separable amenable C^*-algebra. A is said to be *api-weakly stable* with respect to $\mathbf{D}$, if for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ satisfying the following:
For any $B \in \mathbf{D}$ and any positive linear contraction $L : A \to B$ which is $\mathcal{G}$-δ-multiplicative such that $\|L(a)\| \ge (1/2)\|a\|$ for all $a \in \mathcal{G}$, there exists a homomorphism $h : A \to B$ such that

$$h \approx_\varepsilon L \quad \text{on } \mathcal{F}.$$

(Here "api" stands for approximately injective.)

Definition 2.6. Let A be a separable amenable C^*-algebra. A is said to be *apf-weakly semiprojective* with respect to $\mathbf{D}$, if for any sequence $\{B_n\}$ in $\mathbf{D}$ and any full homomorphism $h : A \to q_\infty(\{B_n\})$, there exists a homomorphism $\psi : A \to l^\infty(\{B_n\})$ such that $\pi \circ \psi = h$.

Here "apf" stands for approximately full. One may state this in the form of "apf-weakly stable" (see 5.8 and 5.9 in [Ln10]).

We conclude this section with the following theorem.

Theorem 2.7. *Let A be a separable unital amenable C^*-algebra and let $\mathbf{D}$ be a class of purely infinite simple C^*-algebras. Then A is api-weakly stable with respect to $\mathbf{D}$ if A is apf-weakly semiprojective with respect to $\mathbf{D}$.*

Proof. Let $\{\mathcal{F}_n\}$ be an increasing sequence of finite subsets of the unit ball of A such that $\bigcup_{n=1}^\infty \mathcal{F}_n$ is dense in the unit ball of A. We may further assume that $\bigcup_{n=1}^\infty (A_+ \cap \mathcal{F}_n)$ is also dense in the set of positive element in the unit ball of A.

Suppose the theorem is false. This means that for some $\varepsilon_0 > 0$ and some finite subset $\mathcal{F}_0 \subset A$ in the unit ball, there exists a sequence of C^*-algebras B_n in $\mathbf{D}$ and a sequence $\{\phi_n\}$ of $\mathcal{F}_n$-$1/n$-multiplicative positive linear contractions from A to B_n with $\|\phi_n(a)\| \geq 1/2\|a\|$ for all $a \in \mathcal{F}_n$ such that

$$\sup_n \{\max\{\|\phi_n(a) - h_n(a)\| : a \in \mathcal{F}_0\} \geq \varepsilon_0$$

for any sequence of homomorphisms $h_n : A \to B_n$. Define $\Phi : A \to l^\infty(\{B_n\})$ by $\Phi(a) = \{\phi_n(a)\}$ and define $\phi : A \to q_\infty(\{B_n\})$ by $\phi(a) = \pi \circ \Phi(a)$ for $a \in A$. Then ϕ is a homomorphism. We claim that ϕ is full. In fact, for any $a \in A_+ \setminus \{0\}$ with $\|a\| = 1$, there is $b \in \mathcal{F}_m \cap A_+$ for some m such that

$$\|b - a\| < (1/8)\|a\| = 1/8.$$

Let $b_n = \phi_n(b)$. Then $\|b_n\| \geq (1/2)\|b\| \geq 3/8$. Since B_n has real rank zero (see [Zh3]), it follows from Lemma 6.4 in [Ln10] that there exists a non-zero projection $e_n \in B_n$ such that $b_n \geq (3/8)e_n$. In B_n, there exists a partial isometry $v_n \in B_n$ such that

$$v_n^* e_n v_n = 1.$$

Thus we obtain an element $z_n \in B_n$ such that $\|z_n\| < 16/3$ and

$$z_n^* \phi_n(a) z_n = 1 \quad n = 1, 2, ...$$

Set $Z = \{z_n\}$ and $z = \pi(Z)$. Then we have

$$z^* \phi(a) z = 1$$

in $q_\infty(\{B_n\})$. By the assumption, there exists a homomorphism $\tilde{h} : A \to l^\infty(\{B_n\})$ such that $\pi \circ \tilde{h} = \phi$. Write $\tilde{h} = \{h_n\}$. Then we have

$$\|\phi_n(a) - h_n(a)\| \to 0 \quad \text{as } n \to \infty$$

for all $a \in A$. This gives a contradiction. □

3. Maps to $\mathcal{O}_2$

Lemma 3.1. *Let A be a separable unital C^*-algebra, let B be a purely infinite simple C^*-algebra and let $h : A \to B$ be a unital monomorphism. Suppose that $\phi : A \to B$ is an amenable completely positive linear map. Then, for any finite subset $\mathcal{F} \subset A$ and $\varepsilon > 0$, there is an isometry $s \in B$ such that*

$$\|s^*h(a)s - \phi(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}.$$

Proof. Set $A_0 = h(A)$. Let $\psi = \phi \circ h^{-1} : A_0 \to B$. Since ϕ is amenable, so is ψ. Fix $\mathcal{F}$ and $\varepsilon > 0$, it is well known that there is completely positive linear map $\bar{\psi} : B \to B$ such that $\|\bar{\psi}\| = \|\psi\|$ and

$$\|\bar{\psi}(b) - \psi(b)\| < \varepsilon/2 \text{ for all } a \in \mathcal{F}_1,$$

where $\mathcal{F}_1 = \{h(a) : a \in \mathcal{F}\}$ (see for example 2.3.13 in [Ln7]). It follows from Proposition 1.7 in [KP] that there is an isometry $s \in B$ such that

$$\|s^*bs - \bar{\psi}(b)\| < \varepsilon/2 \text{ for } b \in \mathcal{F}_1.$$

Therefore

$$\|s^*h(a)s - \phi(a)\| \le \|s^*h(a)s - \bar{\psi}(h(a))\| + \|\bar{\psi}(h(a)) - \phi(a)\| < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $a \in \mathcal{F}$. □

Lemma 3.2. *Let $n > 0$ be an integer. Then there is an isometry $S \in M_n(q_\infty(\mathcal{O}_2))$ with $S^*S = 1_n$ and $SS^* = 1$ satisfying the following: For any separable subspace $E \subset q_\infty(\mathcal{O}_2)$, there are unitaries $u_k \in q_\infty(\mathcal{O}_2)$ such that*

$$Sd_n(a)S^* = \lim_{k\to\infty} u_k^* a u_k$$

for all $a \in E$.

Proof. There is an isometry $s \in M_n(\mathcal{O}_2)$ such that

$$s^*s = 1_n \quad \text{and} \quad ss^* = 1.$$

Define $d_n'(a) = \mathrm{diag}(a, ..., a)$ for $a \in \mathcal{O}_2$ which maps $\mathcal{O}_2$ to $M_n(\mathcal{O}_2)$. Then $sd_n'(a)s^*$ (for $a \in \mathcal{O}_2$) is a homomorphism. It follows [Ro1] that there are unitaries $v_k \in \mathcal{O}_2$ such that

$$sd_n'(a)s^* = \lim_{k\to\infty} v_k^* a v_k$$

for all $a \in \mathcal{O}_2$. Define $S' = (s, s, ..., s, ...)$. So $S' \in M_n(l^\infty(\mathcal{O}_2))$ is an isometry. Let $F = \pi^{-1}(E)$, where $\pi : l^\infty(\mathcal{O}_2) \to q_\infty(\mathcal{O}_2)$ is the quotient map. Since both E and $\bigoplus_{n=1}^\infty \mathcal{O}_2$ are separable, F is separable. Let $\{f_k\}$ be a dense sequence of F. For each k, write $f_k = (f_{k,1}, f_{k,2}, ..., f_{k,m}, ...)$, where $f_{k,m} \in \mathcal{O}_2$, $m, k = 1, 2, ...$ For each k and m there exists $v_{k,m}$ such that

$$\|sd_n(f_{i,m})s^* - v_{k,m}^* f_{i,m} v_{k,m}\| < 1/2^k$$

for $i = 1, 2, ..., k$ and $m = 1, 2, ...$ Put $U_k = (v_{k,1}, v_{k,2}, ...v_{k,m}, ...) \in l^\infty(\mathcal{O}_2)$. Note that U_k is a unitary. Then we have

$$S'd_n''(a)(S')^* = \lim_{k\to\infty} U_k^* a U_k$$

for all $a \in F$, where $d_n'' : l^\infty(\mathcal{O}_2) \to M_n(l^\infty(\mathcal{O}_2)$ is defined by $d_n''(a) = \mathrm{diag}(a, ..., a)$. Now set $S = \pi(S')$ and $u_k = \pi(U_k)$, $k = 1, 2, ...$ We conclude that

$$Sd_n(a)S^* = \lim_{k\to\infty} u_k^* a u_k \quad \text{for all } a \in E.$$ □

Lemma 3.3. *Let A be a unital separable C^*-algebra and $h : A \to q_\infty(\mathcal{O}_2)$ be a unital full monomorphism. Suppose that $\psi : A \to q_\infty(\mathcal{O}_2)$ is a unital amenable completely positive linear map. Then there exists a sequence of isometries $s_n \in q_\infty(\mathcal{O}_2)$ such that*

$$\|s_n^* h(a) s_n - \psi(a)\| \to 0 \text{ as } n \to \infty$$

for all $a \in A$.

Proof. Let $\mathcal{F}$ be a finite subset of A which contains the identity of A. It suffices to show that, for any $1/2 > \varepsilon > 0$, there exists an isometry $T \in q_\infty(\mathcal{O}_2)$ such that

$$T^* h T \approx_\varepsilon \psi \text{ on } \mathcal{F}.$$

Let $\varepsilon/4 > \delta > 0$. Since h is full, by 5.4.1 in [Ln7], there is an integer $n > 0$ and contraction $S \in M_n(q_\infty(\mathcal{O}_2))$ such that

$$\operatorname{ad} S_1^* \circ d_n \circ h \approx_\delta \psi \text{ on } \mathcal{F}.$$

Since $1_A \in \mathcal{F}$ and ψ is unital, we have

$$\|S_1 S_1^* - 1_{q_\infty(\mathcal{O}_2)}\| < \delta.$$

Let $S = (S_1 S_1^*)^{-1/2} S_1$. Then $SS^* = 1$. With a sufficiently small δ, we have

$$\operatorname{ad} S^* \circ d_n \circ h \approx_{\varepsilon/2} \psi \text{ on } \mathcal{F}.$$

It follows from 3.2 that there exists an isometry $V \in q_\infty(\mathcal{O}_2)$ with $V^*V = 1_n$ and $VV^* = 1$ such that

$$\operatorname{ad} V^* \circ d_n \circ h \approx_{\varepsilon/2} h \text{ on } \mathcal{F}.$$

Then

$$d_n \circ h \approx_{\varepsilon/2} \operatorname{ad} V \circ h \text{ on } \mathcal{F}.$$

Hence

$$\operatorname{ad}(SV^*)^* h \approx_{\varepsilon/2} \operatorname{ad} S^* \circ d_n \circ h \approx_{\varepsilon/2} \psi \text{ on } \mathcal{F}.$$

Put $T = VS^*$. Then $T^*T = (VS^*)^*(VS^*) = S(V^*V)S^* = SS^* = 1$. So T is an isometry in $q_\infty(\mathcal{O}_2)$. □

One should compare the following with Lemma 2.2 in [KP]. The point of the following lemma is that the lifting map can be chosen as a monomorphism.

Theorem 3.4. *Let A be a separable unital exact C^*-algebra and suppose that there is a unital full monomorphism $h : A \to q_\infty(\mathcal{O}_2)$ with a unital, completely positive lift $\psi : A \to l^\infty(\mathcal{O}_2)$, i.e., the diagram*

$$\begin{array}{ccc} & & l^\infty(\mathcal{O}_2) \\ & \psi \nearrow & \downarrow \pi \\ A & \xrightarrow{h} & q_\infty(\mathcal{O}_2) \end{array}$$

commutes. Then there is a unital monomorphism $\tilde{h} : A \to l^\infty(\mathcal{O}_2)$ such that $\pi \circ \tilde{h} = h$, i.e., the diagram

$$\begin{array}{ccc} & & l^\infty(\mathcal{O}_2) \\ & \tilde{h} \nearrow & \downarrow \pi \\ A & \xrightarrow{h} & q_\infty(\mathcal{O}_2) \end{array}$$

commutes.

Moreover, if, in addition, A is assumed to be simple, the above also holds without assuming h is full.

Proof. We first prove the case in which h is full.

Write $\psi(a) = (\psi_1(a), \psi_2(a), ...)$, where $\psi_k : A \to \mathcal{O}_2$ are unital completely positive linear maps. It follows from 2.8 in [KP] that there is unital embedding $j_0 : A \to \mathcal{O}_2$. Let $J_0(a) = (j_0(a), j_0(a), ...)$ for $a \in A$. Let $J = \pi \circ J_0$, where $\pi : l^\infty(\mathcal{O}_2) \to q_\infty(\mathcal{O}_2)$ is the quotient map. It is easy to check that J is full. It follows from 3.2 that there is a sequence of isometries $s_n, t_n \in q_\infty(\mathcal{O}_2)$ such that

$$\|s_n^* h(a) s_n - J(a)\| \to 0 \quad \text{and} \quad \|t_n^* J(a) t_n - h(a)\| \to 0$$

as $n \to \infty$ for all $a \in A$.

Let $\imath : \mathcal{O}_2 \to \mathcal{O}_2 \otimes \mathcal{O}_2$ be given by $\imath(x) = x \otimes 1$ and let $\lambda : \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$ be an isomorphism. The existence of such λ is given by [Ro3]. Define $\tilde{\imath} : l^\infty(\mathcal{O}_2) \to l^\infty(\mathcal{O}_2 \otimes \mathcal{O}_2)$ by $\tilde{\imath}((a_1, a_2, ...)) = (\imath(a_1), \imath(a_2), ...)$ for $a \in \mathcal{O}_2$ and define $\tilde{\lambda} : l^\infty(\mathcal{O}_2 \otimes \mathcal{O}_2) \to l^\infty(\mathcal{O}_2)$ by $\tilde{\lambda}((a_1, a_2, ...)) = (\lambda(a_1), \lambda(a_2), ...)$ for $a \in A$. It follows from [Ro1] that

$$\lambda \circ \imath \overset{u}{\sim} \mathrm{id}_{\mathcal{O}_2}.$$

Since A is separable, so are $\psi(A)$ and $J_0(A)$. Thus we have

$$\tilde{\lambda} \circ \tilde{\imath} \circ \psi \overset{u}{\sim} \psi \quad \text{and} \quad \tilde{\lambda} \circ \tilde{\imath} \circ J_0 \overset{u}{\sim} J_0.$$

Since $\tilde{\imath}$ maps $c_0(\mathcal{O}_2)$ into $c_0(\mathcal{O}_2 \otimes \mathcal{O}_2)$ and $\tilde{\lambda}$ maps $c_0(\mathcal{O}_2 \otimes \mathcal{O}_2)$ onto $c_0(\mathcal{O}_2)$, we obtain two induced maps $\bar{\imath} : q_\infty(\mathcal{O}_2) \to q_\infty(\mathcal{O}_2 \otimes \mathcal{O}_2)$ and $\bar{\lambda} : q_\infty(\mathcal{O}_2 \otimes \mathcal{O}_2) \to q_\infty(\mathcal{O}_2)$. It is easy to check that both maps are injective. Since $\tilde{\lambda}$ is surjective, so is $\bar{\lambda}$, whence $\bar{\lambda}$ is an isomorphism. With the notation above, from above, we have

$$\bar{\lambda} \circ \bar{\imath} \circ h \overset{u}{\sim} h \quad \text{and} \quad \bar{\lambda} \circ \bar{\imath} \circ J \overset{u}{\sim} J.$$

Let $C = \{\{1 \otimes a_n\} : \{a_n\} \in l^\infty(\mathcal{O}_2)\}$. Then every element in $\bar{\lambda}(C)$ commutes with $\bar{\lambda} \circ \bar{\imath}(q_\infty(\mathcal{O}_2))$. Define $j : \mathcal{O}_2 \to C$ by $j(a) = (1 \otimes a, 1 \otimes a, ...)$. Then $\pi \circ j : \mathcal{O}_2 \to \bar{\lambda}(C)$ is a unital embedding. It follows from Lemma 6.3.7 in [Ro4] that

$$\bar{\lambda} \circ \bar{\imath} \circ h \overset{u}{\sim} \bar{\lambda} \circ \bar{\imath} \circ J \overset{u}{\sim} J.$$

Hence

$$h \overset{u}{\sim} J.$$

Let $\{a_k\}$ be a dense sequence of A. We obtain a sequence of unitaries $\{U_k\}$ in $q_\infty(\mathcal{O}_2)$ such that

$$\|h(a_i) - U_k^* J(a_i) U_k\| < 1/2^k, \;\; i = 1, 2, ..., k.$$

There is, for each k, a sequence of unitaries $u(n,k) \in \mathcal{O}_2$ such that $\{u(n,k)\} = U_k$. For each k, from the above inequality, we obtain $m(k)$ such that

$$\|\psi_n(a_i) - u(n,k)^* j_0(a_i) u(n,k)\| < 1/2^k, \;\; i = 1, 2, ..., k,$$

for all $n \geq m(k)$. Then, for any i,

$$\lim_{n\to\infty} \|\psi_n(a_i) - u(m(n), n)^* j_0(a_i) u(m(n), n)\| = 0.$$

Since $\{a_k\}$ is dense in A, we have

$$\lim_{n\to\infty} \|\psi_n(a) - u(m(n), n)^* j_0(a) u(m(n), n)\| = 0$$

for all $a \in A$. Define $\tilde{h}(a) = \{\mathrm{ad}\, u(m(n), n) \circ j_0(a)\}$ for $a \in A$. Then $\tilde{h} : A \to l^\infty(\mathcal{O}_2)$ is an injective homomorphism and $\pi \circ \tilde{h} = h$.

Now we consider the case in which A is a separable unital simple exact C^*-algebra and h is not assumed to be full. Let $p \in q_\infty(\mathcal{O}_2)$ such that $h(1_A) = p$. There is a projection $P = \{p_n\} \in l^\infty(\mathcal{O}_2)$ such that $\pi(P) = p$. Let $\{n_k\}$ be the subsequence of $\mathbb{N}$ such that $p_{n_k} \neq 0$ and $p_m = 0$ if $m \neq n_k$ for any k. Let $Q = \{q_n\} \in l^\infty(\mathcal{O}_2)$ be a projection such that $q_m = 1$ if $m = n_k$ for some k otherwise $q_m = 0$. Put $l^\infty_{\{n_k\}}(\mathcal{O}_2) = Ql^\infty(\mathcal{O}_2)Q$ and $\pi(Ql^\infty(\mathcal{O}_2)Q) = q_\infty^{(\{n_k\})}(\mathcal{O}_2)$. We may view h as a homomorphism from A to $q_\infty^{(\{n_k\})}(\mathcal{O}_2)$. We now claim that h is full in $q_\infty^{(\{n_k\})}(\mathcal{O}_2)$. For any $a \in A_+ \setminus \{0\}$, there are $x_1, ..., x_k$ in A such that

$$\sum_{i=1}^k x_i^* a x_i = 1_A.$$

Thus

$$\sum_{i=1}^k h(x_i)^* h(a) h(x_i) = p.$$

In $\mathcal{O}_2$, there are partial isometries v_k, such that $v_k^* p_{n_k} v_k = 1$. Set $V = \{v_{n_k}\}$ and $w = \pi(V)$. Then we have

$$\sum_{i=1}^k w^* h(x_i) h(a) h(x_i) w = 1$$

in $q_\infty^{(\{n_k\})}(\mathcal{O}_2)$. This proves that h is full in $q_\infty^{(\{n_k\})}(\mathcal{O}_2)$. It now follows from what we have proved that there is a homomorphism $\tilde{h}' : A \to l^\infty_{\{n_k\}}(\mathcal{O}_2)$ such that $\pi \circ \tilde{h}' = h$. Write $\tilde{h}' = \{h_{n_k}\}$, where each h_{n_k} is a homomorphism from A to $\mathcal{O}_2$. We now define $\tilde{h} : A \to l^\infty(\mathcal{O}_2)$ by $\tilde{h} = \{h_m\}$, where $h_m = h_{n_k}$ if $m = n_k$ and $h_m = 0$ otherwise. □

Corollary 3.5. *Let A be a unital separable simple amenable C^*-algebra. Then A is weakly semiprojective with respect to $\mathcal{O}_2$.*

Corollary 3.6. *Let A be a unital separable amenable C^*-algebra. Then A is apiweakly stable with respect to $\mathcal{O}_2$.*

Proof. This follows from Theorem 2.7 and Theorem 3.4. □

4. Approximately multiplicative maps

Definition 4.1. Let A and B be two C^*-algebras and let $\psi_n : A \to B$ be a sequence of maps from A to B. We say that $\{\psi_n\}$ is *approximately linear* if

$$\lim_{n\to\infty} \|[\alpha\psi_n(a) + \beta\psi_n(b)] - \psi_n(\alpha a + \beta b)\| = 0 \text{ for all } a, b \in A \quad \text{and} \quad \alpha, \beta \in \mathbb{C},$$

is *approximately selfadjoint* if

$$\lim_{n\to\infty} \|\psi_n(a^*) - \psi_n(a)^*\| = 0 \text{ for all } a, b \in A$$

and is *approximately multiplicative* if

$$\lim_{n\to\infty} \|\psi_n(a)\psi_n(b) - \psi_n(ab)\| = 0 \text{ for all } a, b \in A,$$

respectively.

Definition 4.2. Let $\phi_n, \psi_n : A \to B$ be two sequences of maps. We say that $\{\phi_n\}$ and $\{\psi_n\}$ are *approximately equal* if

$$\lim_{n\to\infty} \|\phi_n(a) - \psi_n(a)\| = 0 \text{ for all } a \in A.$$

We say $\{\phi_n\}$ and $\{\psi_n\}$ are *approximately unitarily equivalent* if there exists a sequence of unitaries $\{u_n\}$ in $\tilde{B}$ such that

$$\lim_{n\to\infty} \|\mathrm{ad}\, u_n \circ \phi_n(a) - \psi_n(a)\| = 0$$

for all $a \in A$.

We say two sequences $\{\phi_n\}$ and $\{\psi_n\}$ (of approximately multiplicative (amenable) contractive completely positive linear maps) are homotopic if there is a sequence $F_n : A \to C([0,1], B)$ (of approximately multiplicative (amenable) contractive completely positive linear maps) such that $\{\delta_0 \circ F_n\}$ and $\{\phi_n\}$ are approximately equal and $\{\delta_1 \circ F_n\}$ and $\{\psi_n\}$ are approximately equal, where $\delta_0, \delta_1 : C([0,1], B) \to B$ are the point-evaluation at 0 and 1, respectively.

Definition 4.3. *Let D be a σ-unital C^*-algebra. We say D satisfies condition* (P) *if there is a full embedding from $\mathcal{O}_2$ to $D \otimes \mathcal{K}$.*

The following lemma follows from 1.13 in [KP].

Lemma 4.4. *Let A be a separable unital exact C^*-algebra and let D be a C^*-algebra satisfying property* (P). *Let $j : \mathcal{O}_2 \to D \otimes \mathcal{K}$ be the full embedding and let $h_1, h_2 : A \to D \otimes \mathcal{K}$ be two injective homomorphisms such that there are $\psi_i : A \to \mathcal{O}_2$ with $h_i = j \circ \psi_i$, $i = 1, 2$. Then h_1 and h_2 are approximately unitarily equivalent.*

Proof. Since $j : \mathcal{O}_2 \to D \otimes \mathcal{K}$ is a full embedding, $e = j(1_{\mathcal{O}_2})$ is a full projection. It follows from [Br1] that $eDe \otimes \mathcal{K} \cong D \otimes \mathcal{K}$. Therefore there is full embedding from $\mathcal{O}_2 \otimes \mathcal{K}$ into $D \otimes \mathcal{K}$. We may view both ψ_1 and ψ_2 maps A into $\mathcal{O}_2$. Thus there is a unitary $u \in \widetilde{\mathcal{O}_2 \otimes \mathcal{K}} \subset \widetilde{D \otimes \mathcal{K}}$ such that

$$u^* \phi_1(1_{\mathcal{O}_2}) u = \phi_2(1_{\mathcal{O}_2}).$$

Thus we may assume that both ϕ_1 and ϕ_2 are unital. We can then apply 1.13 in [KP]. $\square$

Theorem 4.5. *Let A be a separable unital exact C^*-algebra and $\{\phi_n\}$ be a sequence of approximately multiplicative amenable contractive completely positive linear maps from A to $D \otimes \mathcal{K}$, where D satisfies property* (P). *Let $j : A \to \mathcal{O}_2 \subset D \otimes \mathcal{K}$ be the embedding given by* 2.8 *in* [KP] *and given by property* (P). *Then there is a sequence of approximately multiplicative amenable contractive completely positive linear maps $\{\bar{\phi}_n\}$ from A to $D \otimes \mathcal{K}$ such that there is a sequence of unitaries $u_n \in \widetilde{D \otimes \mathcal{K}}$ satisfying*

$$\|(\phi_n \oplus \bar{\phi}_n)(a) - u_n^* j(a) u_n\| \to 0 \text{ as } n \to \infty$$

for all $a \in A$.

Proof. To simplify notation, without loss of generality, we may assume that $\phi_n(1_A) = e_n$ are projections in $D \otimes \mathcal{K}$. We identify $\mathcal{O}_2$ with its image given by property (P). It follows from [KP] that there is an embedding $j : A \to \mathcal{O}_2$. By replacing $\{\phi_n\}$ by $\{\phi_n \oplus j\}$, we may assume that $\{\phi_n\}$ are full. Let $\mathcal{F}_1 \subset \mathcal{F}_2, \ldots$ be an increasing sequence of finite subsets of A such that its union is dense in A. We may assume that $1_A \in \mathcal{F}_1$. Let ε_n be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$. It follows from 5.4.2 in [Ln7] that there is an integer $k(n) > 0$ and a contraction $V_n \in D \otimes \mathcal{K}$ such that

$$\|\phi_n(a) - V_n^* d_{k(n)} \circ j(a) V_n\| < \varepsilon_n/2 \text{ for all } a \in \mathcal{F}_n.$$

In $M_{k(n)}(\mathcal{O}_2)$ there is an isometry v_n such that $v_nv_n^* = 1_{\mathcal{O}_2}$ and (by [Ro1])

$$\|d_{k(n)} \circ j(a) - v_nj(a)v_n^*\| < \varepsilon_n/2.$$

Put $S_n = v_n^*V_n$. Then

$$\|\phi_n(a) - S_n^*j(a)S_n\| < \varepsilon_n$$

for all $a \in \mathcal{F}_n$. Since $1_A \in \mathcal{F}_1$, without loss of generality, we may assume that $S_n^*S_n = e_n$ and $S_nS_n^* \le E = j(1_A)$. Let $q_n = S_nS_n^*$. Then q_n is a projection. Put $p_n = E - q_n$. There is a partial isometry $Z_n \in D \otimes \mathcal{K}$ such that $Z_nZ_n^* = p_n$ and $Z_n^*Z_n$ is orthogonal to e_n. Define $W_n = S_n + Z_n$. Then $W_n^*W_n = e_n + Z_nZ_n^*$ and $W_nW_n^* = E$. Let $\psi_n(a) = W_n^*j(a)W_n$ and $\bar{\phi}_n(a) = Z_n^*j(a)Z_n$ for $a \in A$. Denote by 1 the identity of $\widetilde{D \otimes \mathcal{K}}$. Since E and $W_n^*W_n$ are equivalent in $D \otimes \mathcal{K}$, it is known that $1-E$ is equivalent to $1-W_n^*W_n$ in $\widetilde{D \otimes \mathcal{K}}$. So we obtain a unitary $u_n \in \widetilde{D \otimes \mathcal{K}}$ such that $u_n^*Eu_n = W_n^*W_n$ and $Eu_n = W_n$. Therefore $\psi_n(a) = u_n^*j(a)u_n$ for all $a \in A$. To show that

$$\|\phi_n(a) \oplus \bar{\phi}_n(a) - u_n^*j(a)u_n\| \to 0$$

as $n \to \infty$ for all $a \in A$, it suffices to show that

$$\|q_nj(a) - j(a)q_n\| \to 0 \text{ as } n \to \infty$$

for all $a \in A$. Since

$$\|S_n^*j(a)S_nS_n^*j(b)S_n - S_n^*j(ab)S_n\| \to 0 \text{ as } n \to \infty$$

for all $a, b \in A$, we have

$$\|q_nj(a^*)q_nj(a)q_n - q_nj(a^*)j(a)q_n\| \to 0 \text{ as } n \to \infty$$

for all $a \in A$. Equivalently,

$$\|q_nj(a^*)j(a)p_n\| \to 0 \text{ as } n \to \infty$$

for all $a \in A$. Hence

$$\|q_nj(b)p_n\| \to 0 \text{ as } n \to \infty$$

for all $b \in A_+$. Since $E = j(1_A)$, this implies that

$$\|q_nj(b)(1 - q_n)\| \to 0 \text{ as } n \to \infty$$

for all $b \in A_+$. This implies that

$$\|q_nj(b) - j(b)q_n\| \to 0 \text{ as } n \to \infty$$

for all $b \in A_+$. Therefore

$$\|p_nj(b) - j(b)p_n\| \to 0 \text{ as } n \to \infty$$

for all $b \in A_+$. So it holds for all $b \in A$.

Finally, since j is a homomorphism, we conclude that $\{\bar{\phi}_n(a)\} = \{Z_n^*j(a)Z_n\}$ is approximately multiplicative. □

Theorem 4.6 (cf. 2.3.7 in [P3])**.** *Let A be a unital separable exact C^*-algebra and let $\{\phi_n\}$ and $\{\psi_n\}$ be sequences of approximately multiplicative amenable contractive completely positive linear maps from A to $D \otimes \mathcal{K}$ which satisfy property* (P). *Suppose that $\{\phi_n\}$ and $\{\psi_n\}$ are homotopic. Then $\{\phi_n \oplus j\}$ and $\{\psi_n \oplus j\}$ are approximately unitarily equivalent, where $j : A \to \mathcal{O}_2 \to D \otimes \mathcal{K}$ is as in 4.5.*

Proof. We may assume that there is a sequence $\{F_n\}$ of approximately multiplicative amenable contractive completely positive linear maps from A to $C([0,1], D\otimes\mathcal{K})$ such that $\delta_0 \circ F_n = \phi_n$ and $\delta_1 \circ F_n = \psi_n$, where $\delta_t : C([0,1], D\otimes\mathcal{K}) \to D\otimes\mathcal{K}$ is the point evaluation at t ($t \in [0,1]$). Fix $\varepsilon > 0$ and a finite subset $\mathcal{F} \subset A$. For any sufficiently large n, let $0 = t_0 < t_1 < t_2 < \cdots < t_{k+1} = 1$ such that

$$\|\delta_{t_{i+1}} \circ F_n(a) - \delta_{t_i} \circ F_n(a)\| < \varepsilon/5$$

for all $a \in \mathcal{F}$, $i = 1, ..., k$. Put $l_{n,i} = \delta_{t_i} \circ F_n$, $i = 0, 1, ..., k+1$. Note $l_{n,0} = \phi_n$ and $l_{n,k+1} = \psi_n$. By applying 4.5, we obtain $\bar{l}_{n,i} : A \to D\otimes\mathcal{K}$ such that there are unitaries $u_{n,i} \in \widetilde{D\otimes\mathcal{K}}$ such that

$$l_{n,i} \oplus \bar{l}_{n,i} \approx_{\varepsilon/5} \operatorname{ad} u_{n,i} \circ j \ \text{ on } \ \mathcal{F}.$$

Let

$$L_n = \bar{l}_{n,1} \oplus l_{n,1} \oplus \bar{l}_{n,2} \oplus l_{n,2} \oplus \cdots \oplus \bar{l}_{n,k+1} \oplus l_{n,k+1} \quad \text{and}$$
$$L_n' = l_{n,1} \oplus \bar{l}_{n,1} \oplus l_{n,2} \oplus \bar{l}_{n,2} \oplus \cdots \oplus l_{n,k+1} \oplus \bar{l}_{n,k+1}.$$

Then,

$$l_{n,0} \oplus L_n \approx_{\varepsilon/5} L_n' \oplus l_{n,k+1} \ \text{ on } \ \mathcal{F}.$$

There is a unitary $V_n \in \widetilde{D\otimes\mathcal{K}}$ such that $\operatorname{ad} V_n \circ (l_{n,k+1} \oplus L_n') = L_n' \oplus l_{n,k+1}$. On the other hand,

$$L_n \overset{u}{\sim}_{\varepsilon/5} d_{k+1} \circ j \quad \text{and} \quad L_n' \overset{u}{\sim}_{\varepsilon/5} d_{k+1} \circ j \ \text{ on } \ \mathcal{F}.$$

We also have, by 1.13 in [KP],

$$d_{k+1} \circ j \overset{u}{\sim}_{\varepsilon/5} j \ \text{ on } \ \mathcal{F}.$$

Combining these we have

$$l_{n,0} \oplus j \overset{u}{\sim}_{\varepsilon/5} l_{n,0} \oplus d_{k+1} \circ j \overset{u}{\sim}_{\varepsilon/5} l_{n,0} \oplus L_n \overset{u}{\sim}_{\varepsilon/5} l_{n,k+1} \oplus L_n'$$
$$\overset{u}{\sim}_{\varepsilon/5} l_{n,k+1} \oplus d_{k+1} \circ j \overset{u}{\sim}_{\varepsilon/5} l_{n,k+1} \oplus j \ \text{ on } \ \mathcal{F}.$$

In other words,

$$\phi_n \oplus j \overset{u}{\sim}_{\varepsilon} \psi_n \oplus j \ \text{ on } \ \mathcal{F}.$$

□

Lemma 4.7 (cf. 1.3.7 in [P3])**.** *Let A be a separable C^*-algebra and $\{\phi_n\} : A \to D$ be a sequence of approximately multiplicative contractive completely positive linear maps, where D is a unital C^*-algebra. Suppose that, for any finite subset set $\mathcal{F} \subset A$ and any $\varepsilon > 0$, there is an integer N such that, for all $n > N$, there are unitaries $u_{n,k} \in D$ such that*

$$\|u_{n,k}^* \phi_n(a) u_{n,k} - \phi_k(a)\| < \varepsilon \ \textit{ for all } \ a \in \mathcal{F},$$

$k = n+1, n+2, ...$ Then there is a homomorphism $h : A \to D$, a sequence of unitaries $v_n \in D$ and a subsequence $\{m(n)\}$ such that $\lim_{n\to\infty} \operatorname{ad} v_{m(n)} \circ \phi_{m(n)}(a) = h(a)$ for all $a \in A$.

Proof. Let $\mathcal{F}_1, \mathcal{F}_2, ...$ be an increasing sequence of finite subsets of A such that its union is dense in A. There is, for each j, an integer $m(j)$ such that

$$\operatorname{ad} u_{m(j),k} \circ \phi_{m(j)} \approx_{1/2^{j+1}} \phi_{m(j)+k} \ \text{ on } \ \mathcal{F}_j, \ \ k = 1, 2, ...$$

Set $w_n = u_{m(n), m(n+1)-m(n)}$. Then

$$\operatorname{ad} w_n \circ \phi_{m(n)} \approx_{1/2^n} \phi_{m(n+1)} \ \text{ on } \ \mathcal{F}_j$$

for all $j \le n$. Set

$$v_n = w_1 w_2 \cdots w_n,\ n = 1, 2, \ldots.$$

Thus

$$v_{n+k}^* v_n = w_{n+k-1}^* w_{n+k-2}^* \cdots w_n^*.$$

We claim that $h(a) = \lim_{n\to\infty} v_n \phi_{m(n)} v_n^*(a)$ exists for all $a \in A$. It suffices to show that, for each $a \in \mathcal{F}_j$ (for all j), $\{v_n \phi_{m(n)} v_n^*\}$ is Cauchy. For $n > j$, we have

$$\begin{aligned}\|v_n\phi_{m(n)}(a)v_n^* - v_{n+k}\phi_{m(n+k)}(a)v_{n+k}\| &= \|v_{n+k}^* v_n \phi_{m(n)}(a) v_n^* v_{n+k} - \phi_{m(n+k)}(a)\| \\ &= \|w_{n+k-1}^* w_{n+k-2}^* \cdots w_n^* \phi_n(a) w_n \cdots w_{n+k-2} w_{n+k-1} - \phi_{m(n+k)}(a)\| \\ &\le \sum_{i=0}^{k-1} \|w_{n+i}^* \phi_{m(n+i)}(a) w_{n+i} - \phi_{m(n+i+1)}(a)\| < \sum_{i=0}^{k-1} (1/2)^{j+i+1} < 1/2^j.\end{aligned}$$

This proves the claim. Since $\{\phi_n\}$ is approximately multiplicative, we conclude that h is a homomorphism. □

5. The functor E_A

In this section, we will study the functor E_A (defined in 5.7). These results are similar to the results in a paper of N. C. Phillips ([P3]). In [P3], the asymptotic morphisms are from a separable amenable simple C^*-algebra A to $B^+ \otimes \mathcal{O}_\infty \otimes \mathcal{K}$. We do not assume that A is simple. The equivalence for asymptotic morphisms used in [P3] is the approximately unitary equivalence (or homotopy equivalence). Here we use a much weaker stable equivalence. But the main difference is that we consider a sequence of contractive completely positive linear maps, while in [P3] a family $t \to \phi_t$ of maps was considered. However, all proofs here closely follow from those in [P3]. The main purpose of this section is to prove Theorem 5.18.

Definition 5.1. Let C_n be a commutative C^*-algebra with $K_0(C_n) = \mathbb{Z}/n\mathbb{Z}$ and $K_1(C_n) = 0$. Suppose that A is a C^*-algebra. Then $K_i(A, \mathbb{Z}/k\mathbb{Z}) = K_i(A \otimes C_k)$ (see [S3]). One has the following six-term exact sequence ([S3]):

$$\begin{array}{ccccc} K_0(A) & \to & K_0(A, \mathbb{Z}/k\mathbb{Z}) & \to & K_1(A) \\ \uparrow_{\mathbf{k}} & & & & \downarrow_{\mathbf{k}} \\ K_0(A) & \leftarrow & K_1(A, \mathbb{Z}/k\mathbb{Z}) & \leftarrow & K_1(A)\,. \end{array}$$

As in [DL2], we use the notation

$$\underline{K}(A) = \bigoplus_{i=0,1,n\in\mathbb{Z}_+} K_i(A; \mathbb{Z}/n\mathbb{Z}).$$

There is a second six-term exact sequence (see [S3]):

$$\begin{array}{ccccc} K_0(A, \mathbb{Z}/mk\mathbb{Z}) & \to & K_0(A, \mathbb{Z}/k\mathbb{Z}) & \to & K_1(A, \mathbb{Z}/m\mathbb{Z}) \\ \uparrow & & & & \downarrow \\ K_0(A, \mathbb{Z}/m\mathbb{Z}) & \leftarrow & K_1(A, \mathbb{Z}/k\mathbb{Z}) & \leftarrow & K_1(A, \mathbb{Z}/mk\mathbb{Z})\,. \end{array}$$

By $Hom_\Lambda(\underline{K}(A), \underline{K}(B))$ we mean all homomorphisms from $\underline{K}(A)$ to $\underline{K}(B)$ which respects the direct sum decomposition and the above two six-term exact sequences (see [DL2]). It follows from the definition in [DL2] that if $x \in KK(A, B)$, then the Kasparov product (associated with x) gives a homomorphism

$$\Gamma(x) : Hom_\Lambda(\underline{K}(A), \underline{K}(D)) \to Hom_\Lambda(\underline{K}(B), \underline{K}(D))$$

for any C^*-algebra D. It is shown by Dadarlat and Loring ([DL2]) that if A is in $\mathcal{N}$, then for any σ-unital C^*-algebra B, Γ is surjective and $\ker \Gamma = Pext(K_*(A), K_*(B))$.

Let $0 \to I \to B \to C \to 0$ be a short exact sequence of C^*-algebras. From the six term exact sequences

$$\begin{array}{ccccc} K_0(I) & \to & K_0(B) & \to & K_0(C) \\ \uparrow_{\partial_1} & & & & \downarrow_{\partial_0} \\ K_1(C) & \leftarrow & K_1(B) & \leftarrow & K_1(I) \end{array}$$

one easily obtains the following.

Proposition 5.2. *Let A be a separable C^*-algebra and let*

$$0 \to I \to B \to C \to 0$$

be a sequence of σ-unital C^*-algebras. Then one has the following six-term exact sequence:

$$\begin{array}{ccccc} Hom_\Lambda(\underline{K}(A), \underline{K}(I)) & \to & Hom_\Lambda(\underline{K}(A), \underline{K}(B)) & \to & Hom_\Lambda(\underline{K}(A), \underline{K}(C)) \\ \uparrow_{\tilde{\partial}_1} & & & & \downarrow_{\tilde{\partial}_0} \\ Hom_\Lambda(\underline{K}(A), \underline{K}(C)) & \leftarrow & Hom_\Lambda(\underline{K}(A), \underline{K}(B)) & \leftarrow & Hom_\Lambda(\underline{K}(A), \underline{K}(I)), \end{array}$$

where the map $\tilde{\partial}$ is induced by the index map $\partial_i : K_i(B, \mathbb{Z}/k\mathbb{Z}) \to K_{i+1}(B, \mathbb{Z}/k\mathbb{Z})$, $k = 0, 1, ...$ Moreover, if the short exact sequence splits, then $\tilde{\partial}_i = 0$ and the above six-term exact sequence degenerates into two split exact sequences.

Definition 5.3. Let A and B be two C^*-algebras and let $L : A \to B$ be a contractive completely positive linear map. We will still use L for $L \otimes \mathrm{id}_{M_n} : M_n(A) \to M_n(B)$, $L \otimes \mathrm{id}_{M_n} \otimes \mathrm{id}_{C_n} : M_n(A) \otimes C(C_n) \to M_n(B) \otimes C_n$, its extension from $\tilde{M}_n(A) \otimes C(C_n)$ to $\tilde{M}_n(B) \otimes C(C_n)$ and $L \otimes \mathrm{id}_{M_n} \otimes \mathrm{id}_{C(S^1) \otimes C(C_n)}$ and its unitization. Let $\mathbf{P}$ be the set of projections in $M_n(A)$, $M_n(\tilde{A} \otimes C_n)$ and $M_n(\tilde{A} \otimes C_n \otimes C(S^1))$. As discussed in [Ln5] and other places such as [DE1], given a finite subset $\mathcal{P} \subset \mathbf{P}(A)$, there exists $\delta > 0$ and a finite subset $\mathcal{F}$, such that any $\mathcal{F}$-δ-multiplicative contractive completely positive linear map $L : A \to B$ uniquely defines a map from $[\mathcal{P}]$ to $\underline{K}(B)$. Let G be the group generated by $\mathcal{P}$; with an even larger $\mathcal{F}$ and a smaller δ, L gives a group homomorphism $[L]$ from G to $\underline{K}(A)$. In what follows, for a contractive completely positive linear map $L : A \to B$, whenever we write $[L]|_{\mathcal{P}}$ we mean L is $\mathcal{F}$-δ-multiplicative with sufficiently large $\mathcal{F}$ and sufficiently small δ so that $[L]|_{\mathcal{P}}$ is well defined.

Definition 5.4. Let A be a separable unital amenable C^*-algebra. We fix an embedding $j_o : \mathcal{O}_2 \to \mathcal{O}_\infty$. Let B be another C^*-algebra. We use B^+ for the C^*-algebra obtained by adding an identity to B (even if B has a unit). We identify $j_o : \mathcal{O}_2 \to B^+ \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ with the map $a \mapsto 1 \otimes j_o(a)$. With this identification, j_o is full. Put $B_\infty = B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ and $B^\# = B^+ \otimes \mathcal{O}_\infty \otimes \mathcal{K}$. Fix an embedding $\imath : A \to \mathcal{O}_2$. In what follows, we will use j for $j_o \circ \imath$ whenever it is convenient.

An *asymptotic sequential morphism* $\phi = \{\phi_n\}$ from A to $B^\#$ is a sequence of approximately multiplicative contractive completely positive linear maps $\{\phi_n\}$ from A to $B^\#$ such that there is an element $\alpha \in Hom_\Lambda(\underline{K}(A), \underline{K}(B^\#))$ with the property that

$$[\phi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$$

for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$ and all sufficiently large n.

We say two asymptotic sequential morphisms $\phi = \{\phi_n\}$ and $\psi = \{\psi_n\}$ are *equivalent* if there exists a sequence of unitaries $u_n \in \widetilde{B^\#}$ such that, for all $a \in A$,

$$\|\mathrm{ad}\, u_n \circ (\phi_n \oplus j)(a) - (\psi_n \oplus j)(a)\| \to 0 \text{ as } n \to \infty.$$

We will write $\phi \sim \psi$ if ϕ and ψ are equivalent. Denote by $\{A, B^{\#}\}$ the equivalent classes of asymptotic sequential morphisms from A to $B^{\#}$. If $\phi = \{\phi_n\}$ is an asymptotic sequential morphism from A to $B^{\#}$, we denote by $\langle\phi\rangle$ the equivalence class containing ϕ.

Since $B^{\#} = B^+ \otimes (\mathcal{O}_\infty \otimes \mathcal{K})$, we can add two elements by defining $\langle\phi\rangle + \langle\psi\rangle$ to be $\langle\phi \oplus \psi\rangle$.

Proposition 5.5. *Let $\phi = \{\phi_n\}$ and $\psi = \{\psi_n\}$ be two asymptotic sequential morphisms. If $\phi \sim \psi$, then there is unique $\alpha \in Hom_\Lambda(\underline{K}(A), \underline{K}(B^{\#}))$ such that for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$,*

$$[\phi_n]|_{\mathcal{P}} = [\psi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$$

for all sufficiently large n.

Proof. Since $j : A \to \mathcal{O}_2 \to \mathbb{C} \cdot 1_{B^+} \oplus (\mathcal{O}_\infty \otimes \mathcal{K})$, $[j] = 0$ in $Hom_\Lambda(\underline{K}(A), \underline{K}(B^{\#}))$. Therefore $[\phi_n]|_{\mathcal{P}} = ([\phi_n] \oplus [j])|_{\mathcal{P}}$ and $[\psi_n]|_{\mathcal{P}} = ([\psi_n] \oplus [j])|_{\mathcal{P}}$. Suppose there is an $\alpha \in Hom_\Lambda(\underline{K}(A), \underline{K}(B^{\#}))$ such that, for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$,

$$[\phi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$$

for all large n. Since there exists a sequence of unitaries $\{u_n\}$ in $\widetilde{B^{\#}}$ such that

$$\|\mathrm{ad}\, u_n \circ \phi_n(a) - \psi_n(a)\| \to 0 \ \text{ as } \ n \to \infty$$

for all $a \in A$, it follows that

$$[\phi_n]|_{\mathcal{P}} = [\psi_n]|_{\mathcal{P}}$$

for all large n. □

Proposition 5.6. *$\{A, B^{\#}\}$ is a group.*

Proof. From the definition, it is immediate that j represents the zero element. Let $\phi = \{\phi_n\}$ be an asymptotic sequential morphismfrom A to $B^{\#}$. It follows from 4.5 that there is a sequence of approximately contractive completely positive linear maps $\{\bar{\phi}_n\}$ from A to $B^{\#}$ and a sequence of unitaries $u_n \in \widetilde{B^{\#}}$ such that

$$\|\mathrm{ad} u_n \circ j(a) - (\phi_n \oplus \bar{\phi}_n)(a)\| \to 0 \ \text{ as } \ n \to \infty$$

for all $a \in A$. Suppose that $\alpha \in Hom_\Lambda(\underline{K}(A), \underline{K}(B^{\#}))$ such that, for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$,

$$[\phi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$$

for all large n. Note that $[j] = 0$. Hence

$$([\phi_n] + [\bar{\phi}_n])|_{\mathcal{P}} = 0$$

for all large n. Therefore

$$([\bar{\phi}_n])|_{\mathcal{P}} = -\alpha|_{\mathcal{P}}$$

for all n. Thus $\bar{\phi} = \{\bar{\phi}_n\}$ is an asymptotic sequential morphism from A to $B^{\#}$. This shows that $\{A, B^{\#}\}$ is a group. □

Definition 5.7. Fix a unital separable amenable C^*-algebra A. Let B be a separable C^*-algebra and $\xi : B^{\#} \to \mathcal{O}_\infty \otimes \mathcal{K}$ be given by the unitization. Let $\phi = \{\phi_n\}$ be an asymptotic sequential morphism from A to $B^{\#}$. We denote by $E_A(B)$ those $\langle\phi\rangle$ such that there exists a sequence of unitaries $u_n \in \widetilde{\mathcal{O}_\infty \otimes \mathcal{K}}$ such that

$$\|\mathrm{ad}\, u_n \circ j(a) - \xi \circ \phi_n(a) \oplus j(a)\| \to 0 \ \text{ as } \ n \to \infty$$

for all $a \in A$.

Lemma 5.8. *Let A be a unital separable amenable C^*-algebra. Then $E_A(B)$ is a group for each σ-unital C^*-algebra. Moreover, if $\langle\phi\rangle \in E_A(B)$ is represented by $\{\phi_n\}$, then there is $\alpha \in Hom_\Lambda(\underline{K}(A), \underline{K}(B_\infty))$ such that*

$$[\phi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$$

for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$ and any sufficiently large n.

Proof. Let $\xi : B^\# \to \mathcal{O}_\infty \otimes \mathcal{K}$ be the surjective map associated with the unitization of B. Let $\langle\phi\rangle \in E_A(B)$ which is represented by $\{\phi_n\}$. Let $\langle\bar{\phi}\rangle \in \{A, B^\#\}$ be represented by $\{\bar{\phi}_n\}$. We may assume that, for each $a \in A$, there are $u_n \in U(\widetilde{B^\#})$ such that

$$\|\mathrm{ad}\, u_n \circ \xi \circ j(a) - (\phi_n(a) \oplus \bar{\phi}_n(a)\| \to 0$$

as $n \to \infty$ for all $a \in A$. Since $\langle\phi\rangle \in E_A(B)$, we may assume that there exists a sequence of unitaries $w_n \in U(\widetilde{B^\#})$ such that

$$\|\mathrm{ad}\xi(w_n) \circ \xi \circ j(a) - \xi \circ \phi_n(a)\| \to 0$$

as $n \to \infty$ for all $a \in A$. This implies that

$$\|\mathrm{ad}\, \xi \circ (z_n \circ \mathrm{ad}\, u_n \circ j)(a) - \xi \circ (j(a) \oplus \bar{\phi}_n)(a)\| \to 0$$

as $n \to \infty$ for all $a \in A$, where $z_n = \mathrm{diag}(w_n, 1)$. This shows that $\langle\bar{\phi}\rangle \in E_A(B)$, Thus $E_A(B)$ is a group.

To see that $\alpha \in Hom_\Lambda(\underline{K}(A), \underline{K}(B_\infty))$, we let $\lambda : \mathcal{O}_\infty \otimes \mathcal{K} \to B^\#$ be the map associated with the unitization of B. From the six-term exact sequence in K-theory induced by the split short exact sequence

$$0 \to B \otimes \mathcal{K} \to B^\# \to \mathcal{O}_\infty \otimes \mathcal{K} \to 0$$

and the fact that $K_0(\mathcal{O}_\infty) = \mathbb{Z}$ and $K_1(\mathcal{O}_\infty) = 0$, we compute that

$$\underline{K}(B^\#) = \underline{K}(B_\infty) \oplus \underline{K}(\mathcal{O}_\infty)$$

and

$$Hom_\Lambda(\underline{K}(A), \underline{K}(B^\#)) = Hom_\Lambda(\underline{K}(A), \underline{K}(B_\infty)) \oplus Hom_\Lambda(\underline{K}(A), \underline{K}(\mathcal{O}_\infty)).$$

Since we may assume that

$$\|\xi \circ \phi_n(a) - \mathrm{ad}\xi(u_n) \circ \xi(j)(a)\| \to 0$$

as $n \to \infty$ for all $a \in A$ and $[\xi \circ (\mathrm{ad} u_n \circ j)] = 0$ in $Hom_\Lambda(\underline{K}(A), \underline{K}(\mathcal{O}_\infty))$, we conclude that

$$[\xi] \circ \alpha = 0$$

in $Hom_\Lambda(\underline{K}(A), \underline{K}(\mathcal{O}_\infty))$. Therefore we may write that

$$\alpha \in Hom_\Lambda(\underline{K}(A), \underline{K}(B_\infty)).$$

□

Proposition 5.9. *Let A be a unital separable amenable C^*-algebra. Then E_A is a functor from separable C^*-algebras and $*$-homomorphisms to abelian groups.*

Proof. It follows from the definition and 5.8 that $E_A(B)$ is an abelian group for all σ-unital C^*-algebras. For functoriality, let C be another separable C^*-algebra and let $h : B \to C$ be a homomorphism. We extend it to obtain a homomorphism $\tilde{h} : B^+ \to C^+$. Put $\bar{h} = \tilde{h} \otimes \mathrm{id}_{\mathcal{O}_\infty \otimes \mathcal{K}}$. Therefore $\{\phi_n\} \mapsto \{\bar{h} \circ \phi_n\}$ sends asymptotic sequential morphisms to asymptotic sequential morphisms. Moreover, one checks that $\langle\{\bar{h} \circ \phi_n\}\rangle$ is in $E_A(C)$ if $\langle\{\phi_n\}\rangle$ is in $E_A(B)$. Therefore h induces a homomorphism $h_* : E_A(B) \to E_A(C)$.

If $g : C \to D$ is also a homomorphism, then it is easy to check that $(g \circ h)_* = g_* \circ h_*$. Moreover, it is obvious that $(\mathrm{id}_B)_* = \mathrm{id}_{E_A(B)}$. □

Remark 5.10. Suppose that $\phi = \{\phi_n\}$ is an asymptotic sequential morphism from $B^\#$ to $C^\#$. For any asymptotic sequential morphism $\{\psi_n\}$ which represents an element $\langle\psi\rangle$ in $E_A(B)$, $\{\phi_n \circ \psi_n\}$ gives an element in $E_A(C)$. It follows that $\langle\phi\rangle$ also induces a homomorphism from $E_A(B)$ to $E_A(C)$.

Lemma 5.11 (cf. 3.16 in [P3])**.** *Let A be a unital separable amenable C^*-algebra. Let*

$$0 \to I \xrightarrow{\imath} B \xrightarrow{\pi} B/I \to 0$$

be a short exact sequence of separable C^*-algebras.

(1) *Suppose that the map $[\imath] : Hom_\Lambda(\underline{K}(A), \underline{K}(I_\infty)) \to Hom_\Lambda(\underline{K}(A), \underline{K}(B_\infty))$ is injective. Then the sequence*

$$E_A(I) \xrightarrow{\imath_*} E_A(B) \xrightarrow{\pi_*} E_A(B/I)$$

is exact in the middle.

(2) *If there is a C^*-subalgebra $J \subset B$ such that J is contractible and $I \subset J$, then π_* is injective.*

Proof. It follows from Definition 5.7 that $\pi_* \circ \imath_* = 0$. To show that $\ker(\pi_*) \subset \mathrm{im}(\imath_*)$, we let $\xi : B^\# \to \mathcal{O}_\infty \otimes \mathcal{K}$ and $\lambda : \mathcal{O}_\infty \otimes \mathcal{K} \to B^\#$ be the maps associated with the unitization maps $B^+ \to \mathbb{C}$ and $\mathbb{C} \to B^+$. Denote by $\pi^\#$ the quotient map from $B^\# \to (B/I)^\#$ induced by π. Let $\langle\phi\rangle \in \ker \pi_*$, which is represented by an asymptotic sequential morphism $\phi = \{\phi_n\}$ from A to $B^\#$. Then there exists a sequence of unitaries $v_n \in \widetilde{(B/I)^\#}$ such that

$$\|\mathrm{ad} v_n \circ \pi^\# \circ (\phi_n(a) \oplus j(a)) - j(a)\| \to 0 \text{ as } n \to \infty$$

for all $a \in A$. By replacing v_n by $v_n \oplus v_n^*$, we may assume that $v_n \in U_0(\widetilde{(B/I)^\#})$. Let $u_n \in U_0(\widetilde{B^\#})$ be such that $\pi^\#(u_n) = v_n$. We have

$$\|\pi^\#(u_n^*(\phi_n \oplus j(a))u_n - j(a))\| \to 0 \text{ as } n \to \infty$$

for all $a \in A$. Let $\sigma : (B/I)^\# \to B^\#$ be a continuous (not necessarily linear) cross section of $\pi^\#$ satisfying $\sigma(0) = 0$ (given by [BG]). Define $\psi_n' : A \to B^\#$ by

$$\psi_n'(a) = u_n^*(\phi_n(a) \oplus j(a))u_n - (\sigma \circ \pi^\#)(u_n^*\phi_n(a)u_n - j(a))$$

for $a \in A$. Since

$$\pi^\#(\psi_n'(a) - j(a)) = 0,$$

$\psi_n'(a) \in I^\#$ for all A. Since σ is continuous, we have

$$\lim_{n\to\infty} \|(\sigma \circ \pi^\#)(u_n^*(\phi_n(a) \oplus j(a))u_n - j(a))\| = 0$$

for all $a \in A$. Therefore $\{\psi_n'\}$ is an approximately linear, self adjoint and multiplicative map (not necessarily linear or positive) from A to $I^\#$. Since A is amenable, it follows from 1.1.5 in [P3] that there is a sequence of approximately multiplicative contractive completely positive linear maps $\psi_n : A \to I^\#$ such that

$$\|\psi_n(a) - \psi_n'(a)\| \to 0 \text{ as } n \to \infty$$

for all $a \in A$. Therefore

$$\lim_{n\to\infty} \|\imath \circ \psi_n(a) - \mathrm{ad}\, u_n \circ (\phi_n(a) \oplus j(a))\| = 0$$

for all $a \in A$.

If there is a C^*-subalgebra $J \subset B$ such that J is contractible and $I \subset J$, then it follows from 4.6 that there are unitaries $z_n \in \widetilde{B^\#}$ such that

$$\lim_{n\to\infty} \|\mathrm{ad}\, z_n \circ j(a) - \imath \circ \psi_n(a)\| = 0$$

for all $a \in A$. Combining the above two limits, we see that $\langle \phi \rangle = 0$. This implies that π_* is injective. This proves (2).

To prove (1), let $\alpha \in Hom_\Lambda(\underline{K}(A), \underline{K}(B_\infty))$ such that, for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$,

$$[\phi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$$

for all sufficiently large n. Since the map

$$[\imath] : Hom_\Lambda(\underline{K}(A), \underline{K}(I_\infty)) \to Hom_\Lambda(\underline{K}(A), \underline{K}(B_\infty))$$

is injective, it has a left inverse $[\imath]^{-1}$. Then, one checks that

$$[\psi_n]|_{\mathcal{P}} = [\imath]^{-1} \circ \alpha|_{\mathcal{P}}$$

for all sufficiently large n. So $\{\psi_n\}$ is an asymptotic sequential morphism and $\langle\{\psi_n\}\rangle \in E_A(I)$. Moreover $\langle\{\imath \circ \psi_n\}\rangle = \langle \phi \rangle$ and $\langle\{\psi_n\}\rangle \in \mathrm{im}\imath_*$. This completes the proof. □

Lemma 5.12. *Let A be a separable amenable C^*-algebra and let B be a σ-unital C^*-algebra. Suppose that $f_{i*} : E_A(B) \to E_A(C)$ $(i = 1, 2)$ are homomorphisms and there is a homomorphism $g_* : E_A(B) \to E_A(C([0, 1], C))$ such that $\delta_0 \circ g_* = f_{1*}$ and $\delta_1 \circ g_* = f_{2*}$, where $\delta_t : C([0, 1], B) \to B$ is the point evaluation at t. Then $f_{1*} = f_{2*}$.*

Proof. Let $\langle \phi \rangle \in E_A(B)$. Suppose that $g_*(\langle \phi \rangle)$ is represented by $\{\Phi_n\}$, where $\Phi_n : A \to C([0, 1], B)$ is a sequence of contractive completely positive linear maps. Then $\delta_t \circ g_*(\langle \phi \rangle)$ is represented by $\{\delta_t \circ \Phi_n\}$. It follows from 4.6 that $\langle\{\delta_0 \circ \Phi_n\}\rangle = \langle\{\delta_1 \circ \Phi_n\}\rangle$. However, $f_{0*}(\langle \phi \rangle) = \langle\{\delta_0 \circ \Phi_n\}\rangle$ and $f_{1*}(\langle \phi \rangle) = \langle\{\delta_1 \circ \Phi_n\}\rangle$. Therefore $f_{1*} = f_{2*}$. □

Lemma 5.13. *Let A be a separable amenable C^*-algebra and*

$$0 \to I \xrightarrow{\imath} B \xrightarrow{\pi} B/I \to 0$$

be a split short exact sequence of separable C^*-algebras. Then we have the following split short exact sequence:

$$0 \to E_A(I) \xrightarrow{\imath_*} E_A(B) \xrightarrow{\pi_*} E_A(B/I) \to 0.$$

Proof. We first show that if B/I is contractible, then the embedding $\imath : I \to B$ gives an isomorphism $E_A(I)$ to $E_A(B)$. By 5.12, $E_A(B/I) = 0$. Moreover, the map $\tilde{\partial}_1 : Hom_\Lambda(\underline{K}(A), \underline{K}(B/I)) \to Hom_\Lambda(\underline{K}(A), \underline{K}(I))$ in 5.2 is zero. Therefore $[\imath]$ is injective. It follows from 5.11 that i_* is surjective.

We also have $E_A(B/I) = E_A(S(B/I)) = E_A(C_0((0, 1], I)) = 0$. Set

$$S(B, B/I) = \{(a, b) \in B \oplus C_0([0, 1), B/I) : \pi(a) = b(0)\} \quad \text{and}$$

$$Z(I, B) = \{x \in C([0, 1], B) : x(0) \in I\}.$$

Since $S(B/I)$ and $C_0((0,1], I)$ are contractible, by 5.11 we have exact sequences

$$0 = E_A(S(B/I)) \to E_A(B, B/I)) \to E_A(B) \quad \text{and}$$
$$0 = E_A(C_0(0,1], I) \to E_A(Z(I,B)) \to E_A(S(B,B/I)).$$

It follows that $E_A(Z(I,B)) \to E_A(S(B,B/I)) \to E_A(B)$ are monomorphisms. However, $Z(I,B)$ is homotopically equivalent to I, and the composition $I \to Z(I,B) \to S(B,B/I) \to B$ coincides with $\imath$. By 5.12, $\imath_*$ is injective.

In general, let $j : I \to S(B,B/I)$ be defined by $j(b) = (b,0)$. Then $S(B,B/I)/j(I) \cong C_0([0,1), B/I)$ is contractible. So from what we have just proved, j_* is an isomorphism.

To see that E_A is split exact, we assume that $s : B/I \to B$ is a monomorphism such that $\pi \circ s = \mathrm{id}_{B/I}$. It follows from 5.2 and 5.11 that $E_A(I) \to E_A(B) \to E_A(B/I)$ is exact in the middle. Since $\pi_* \circ i_* = (\mathrm{id}_{B/I})_*$, we see that π_* is surjective. It remains to show that $\imath_*$ is injective.

Denote by $j_1 : S(B/I) \to S(B,B/I)$ the embedding. Let

$$J = \{(s(b(0)), b) \in S(B,B/I) : b \in C_0([0,1), B/I)\}.$$

Then $J \cong C_0([0,1), B/I)$, which is contractible. On the other hand, we see that $\mathrm{im} j_1 \subset J$. Thus $(j_1)_* = 0$. Moreover, by applying (2) in 5.11 to the short exact sequence

$$0 \to S(B/I) \to S(B,B/I) \to B \to 0,$$

we see that the map $E_A(S(B,B/I)) \to E_A(B)$ is injective. We again note that $Z(I,B)$ is homotopically equivalent to I, and the composition $I \to Z(I,B) \to S(B,B/I) \to B$ coincides with $\imath$. By 5.12, $\imath_*$ is also injective. $\square$

Let D be a C^*-algebra. Define $\tilde{e} : D \otimes \mathcal{K} \to D \otimes \mathcal{K} \otimes \mathcal{K}$ by $\tilde{e}(a \otimes b) = a \otimes b \otimes e_{11}$, where e_{11} is a rank one projection in $\mathcal{K}$. The map $\tilde{e}$ induces a map $\tilde{e}_* : E_A(B) \to E_A(B \otimes \mathcal{K})$. It should be noted that e_{11} may be chosen to be any rank one projection in $\mathcal{K}$.

Lemma 5.14 (cf. 3.1.11 in [P3]). *Let A be a separable unital amenable C^*-algebra. Then $E_A(-)$ is stable, i.e., $\tilde{e}_* : E_A(B) \to E_A(B \otimes \mathcal{K})$ is an isomorphism.*

Proof. Fix a rank one projection e_{11} in $\mathcal{K}$. Let $\phi : \mathcal{O}_\infty \to \mathcal{K} \otimes \mathcal{O}_\infty$ be the homomorphism defined by $\phi(a) = e_{11} \otimes a$ for $a \in \mathcal{O}_\infty$. Fix an approximate identity $\{e_n\}$ of $\mathcal{K} \otimes \mathcal{O}_\infty$ which consists of projections with $e_1 = e_{11} \otimes 1_{\mathcal{O}_\infty}$. In $\mathcal{K} \otimes \mathcal{O}_\infty$ there is for each n a projection $d_n \geq e_n$ such that $[d_n] = [e_1]$. Note that, for any $b \in \mathcal{K} \otimes \mathcal{O}_\infty$,

$$\|d_n b - b\| \to 0, \quad \|b d_n - b\| \to 0 \quad \text{and} \quad \|d_n b d_n - b\| \to 0$$

as $n \to \infty$. There exists a unitary $u_n \in \widetilde{\mathcal{K} \otimes \mathcal{O}_\infty}$ such that $u_n d_n u_n^* = e_1$, $n = 1, 2, ...$ Define $\psi_n^{(0)}(b) = u_n d_n a d_n u^*$ for all $a \in \mathcal{K} \otimes \mathcal{O}_\infty$, $n = 1, 2, ...$ Then $\{\psi_n^{(0)}\}$ is an asymptotic sequential morphism from $\mathcal{K} \otimes \mathcal{O}_\infty$ to $e_1(\mathcal{K} \otimes \mathcal{O}_\infty)e_1$. So there is an asymptotic sequential morphism $\{\psi_n\}$ from $\mathcal{K} \otimes \mathcal{O}_\infty$ to $\mathcal{O}_\infty$ such that $\phi \circ \{\psi_n\} = \{\psi_n^{(0)}\}$. Note that

$$\mathrm{ad}\, u_n \circ \psi_n^{(0)}(a) = d_n a d_n$$

for all n and $a \in \mathcal{K} \otimes \mathcal{O}_\infty$ which is equivalent to $\mathrm{id}_{\mathcal{K} \otimes \mathcal{O}_\infty}$. Put $\psi = \{\psi_n\}$.

Define $\tilde{e} : B \otimes \mathcal{O}_\infty \otimes \mathcal{K} \to (B \otimes \mathcal{K}) \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ by $\tilde{e}(b \otimes a \otimes k) = b \otimes e_{11} \otimes a \otimes k$ and let $\tilde{e}^\# : B^\# \to (B \otimes \mathcal{K})^\#$ be the extension. We note that $\tilde{e} = \mathrm{id}_B \otimes \phi \otimes \mathrm{id}_{\mathcal{K}}$. Define $\tilde{f} = \mathrm{id}_B \otimes \psi \otimes \mathrm{id}_{\mathcal{K}}$. We see that $\tilde{e}$ and $\tilde{f}$ induce homomorphisms from $E_A(B)$

to $E_A(B\otimes\mathcal{K})$ and from $E_A(B\otimes\mathcal{K})$ to $E_A(B)$. The fact that $\phi\circ\psi$ is equivalent to $\mathrm{id}_{\mathcal{K}\otimes\mathcal{O}_\infty}$ implies that $\tilde{e}\circ\tilde{f}$ induces the identity map from $E_A(B\otimes\mathcal{K})$ to itself.

Consider $\psi\otimes\mathrm{id}_{\mathcal{K}}\circ\phi\otimes\mathrm{id}_{\mathcal{K}}$. For each $a\in\mathcal{O}_\infty$, since $e_n\ge e_1$, one has

$$\psi_n\circ\phi(a)=u_n(e_{11}\otimes a)u_n^*.$$

By identifying $e_{11}\otimes a$ with a, we may view $s_n=u_ne_1$ as an isometry in $\mathcal{O}_\infty$. There is a unitary $W_n\in\widetilde{\mathcal{O}_\infty\otimes\mathcal{K}}$ such that $W_n(1\otimes e_{11})=s_n$. Therefore one sees that $\psi\circ\phi$ induces an isomorphism on $E_A(B)$. It follows that $\tilde{e}_*$ is an isomorphism. $\square$

We summarized the main results of this section as follows:

Theorem 5.15. *Let A be a unital separable amenable C^*-algebra. Then E_A is a functor from the category of separable C^*-algebras (with usual homomorphisms) to the category of abelian groups which is*

(i) *homotopy invariant, i.e., suppose that $f_{i*}:E_A(B)\to E_A(C)$ $(i=1,2)$ are homomorphisms and there is a homomorphism $g_*:E_A(B)\to E_A(C([0,1],C))$ such that $\delta_0\circ g_*=f_{1*}$ and $\delta_1\circ g_*=f_{2*}$, where $\delta_t:C([0,1],B)\to B$ is the point evaluation at t, then $f_{1*}=f_{2*}$;*

(ii) *stable, i.e., $\tilde{e}_*:E_A(B)\to E_A(B\otimes\mathcal{K})$ is an isomorphism; and*

(iii) *split exact.*

Definition 5.16. Let A be a separable amenable C^*-algebra. We use the identification $KK(A,B)=Ext(SA,B)$ and $KK^1(A,B)=Ext(A,B)$. We denote by $\mathcal{T}(A,B)$ the set of equivalence classes of stably approximately trivial extensions (see [Ln10]). It was shown that $\mathcal{T}$ is a subgroup of $Ext(A,B)$.

Let A be a separable amenable C^*-algebra and let B be a σ-unital C^*-algebra. Recall that $KL(A,B)=KL^0(A,B)=Ext(SA,B)/\mathcal{T}(SA,B)$ and $KL^1(A,B)=Ext(A,B)/\mathcal{T}(A,B)$ (see [Ln10]). We will use $\Pi:KK(A,B)\to KL(A,B)$ for the quotient map. It should be noted that we have now defined $KL(A,B)$ without the UCT (see [Ln10]). It follows from [Ln10] that (with A amenable) $\mathcal{T}(A,B)$ is in the kernel of Γ. Thus we obtain the induced map $\tilde{\Gamma}$ from $KL(A,B)$ to $Hom_\Lambda(\underline{K}(A),\underline{K}(B))$.

Recall (see [Ln10]) that A is said to satisfy the Approximate Universal Coefficient Theorem (AUCT) if the map $\tilde{\Gamma}$ is an isomorphism. It is equivalent to have the following exact sequence:

$$0\to Pext_{\mathbb{Z}}(K_*(A),K_*(B))\to KL(A,B)\to Hom(K_*(A),K_*(B))\to 0.$$

It is known and it is easy to see that $KL(A,-)$, $Hom_\Lambda(\underline{K}(A),\underline{K}(-))$, $KL(A,-\otimes\mathcal{O}_\infty)$ and $Hom_\Lambda(A,-\otimes\mathcal{O}_\infty)$ are functors from category of C^*-algebras with $*$-homomorphisms to abelian groups. It is known that both $KL(A,-)$ and $Hom_\Lambda(\underline{K}(A),\underline{K}(-))$ are homotopy invariant, stable and split exact. From this fact, it is rather routine to check that the latter two are also homotopy invariant, stable and split exact.

Definition 5.17. Let A be a unital separable amenable C^*-algebra and let B be a separable C^*-algebra. Given $\langle\phi\rangle\in E_A(B)$, which is represented by an asymptotic sequential morphism $\{\phi_n\}$ from $A\to B^{\#}$, it follows from 5.5 and 5.8 that there is a unique $\alpha\in Hom_\Lambda(\underline{K}(A),\underline{K}(B_\infty))$ such that

$$[\phi_n]|_{\mathcal{P}}=\alpha|_{\mathcal{P}}$$

for all finite subsets $\mathcal{P} \subset \mathbf{P}$ and all sufficiently large n. Let $\beta_B(\langle\phi\rangle) = \alpha$. Then β_B gives a homomorphism from $E_A(B)$ to $Hom_\Lambda(\underline{K}(A), \underline{K}(B \otimes \mathcal{O}_\infty))$. This defines a natural transformation β from the functor E_A to the functor

$$Hom_\Lambda(\underline{K}(A), \underline{K}(- \otimes \mathcal{O}_\infty)).$$

To see this, let $h : B \to D$ be a homomorphism and $\langle\phi\rangle = \langle\{\phi_n\}\rangle \in E_A(B)$. Suppose that $\xi = \beta_B(\langle\phi\rangle)$. Then $h_*(\xi) = \beta_D(\{h \circ \phi_n\})$. In other words, we have

$$\beta_D(E_A(h))(\langle\phi\rangle) = \beta_D(\langle\{h \circ \phi_n\}\rangle) = h_*(\beta_B(\langle\phi_n\rangle)).$$

The composition $KL(A, -) \to KL(A, - \otimes \mathcal{O}_\infty) \to Hom_\Lambda(\underline{K}(A), \underline{K}(- \otimes \mathcal{O}_\infty))$ will be denoted by $\tilde{\Gamma}_1$. Then $\tilde{\Gamma}_1$ is a natural transformation from the functor $KL(A, -)$ to the functor $Hom_\Lambda(\underline{K}(A), \underline{K}(- \otimes \mathcal{O}_\infty))$.

Theorem 5.18. *Let A be a unital separable amenable C^*-algebra. Then, for each separable C^*-algebra B, the image of map β_B contains $\tilde{\Gamma}_1(KL(A, B)) = \tilde{\Gamma}(KL(A, B_\infty))$.*

Proof. It is easy to see that $\tilde{\Gamma}$ is a natural transformation from the functor $KK(A, -)$ to the functor $Hom_\Lambda(\underline{K}(A), \underline{K}(- \otimes \mathcal{O}_\infty))$. It follows from a result of Higson [H] that there is a unique natural transformation α from $KK(A, -)$ to E_A with $\alpha_A([\mathrm{id}_A]) = \langle \mathrm{id}_A\rangle$. Let $\beta : E_A \to Hom_\Lambda(\underline{K}(A), \underline{K}(- \otimes \mathcal{O}_\infty))$ be the natural transformation defined in 5.17. Then $\beta_A(\langle \mathrm{id}_A\rangle) = [\mathrm{id}_A]$. Therefore $\beta_A \circ \alpha_A([\mathrm{id}_A]) = [\mathrm{id}_A]$. Since $\tilde{\Gamma}([\mathrm{id}_A]) = [\mathrm{id}_A]$, it follows from [H] that

$$\tilde{\Gamma}_1 \circ \Pi = \beta \circ \alpha.$$

Thus $\beta_B(E_A(B)) \supset \tilde{\Gamma}_1(KL(A, B)) = \tilde{\Gamma}(KL(A, B \otimes \mathcal{O}_\infty))$. □

Corollary 5.19. *Let A be a unital separable amenable C^*-algebra and let B be a unital separable C^*-algebra. Then for any $\alpha \in \tilde{\Gamma}_1(KL(A, B))$, there exists an asymptotic sequential morphism $\{\phi_n\}$ which maps A into B_∞ such that, for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$,*

$$[\phi_n]|_\mathcal{P} = \alpha|_\mathcal{P}$$

for all sufficiently large n.

Proof. By 5.18 there is an element $\langle\{\phi_n\}\rangle \in E_A(B)$ such that $[\langle\{\phi_n\}\rangle] = \alpha$. Since B is unital, we may write $B^\# = B \otimes \mathcal{O}_\infty \otimes \mathcal{K} \oplus \mathcal{O}_\infty \otimes \mathcal{K}$. Therefore we may write $\phi_n = \phi'_n \oplus \phi''_n$, where $\phi'_n : A \to B_\infty$ and $\phi''_n : A \to \mathcal{O}_\infty \otimes \mathcal{K}$. Moreover both $\{\phi'_n\}$ and $\{\phi''_n\}$ are asymptotic sequential morphisms and there are unitaries $u_n \in \widetilde{\mathcal{O}_\infty \otimes \mathcal{K}}$ such that

$$\|\mathrm{ad}\, u_n \circ \phi''_n(a) - j(a)\| \to 0$$

as $n \to \infty$ for all $a \in A$. It follows that, for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$,

$$[\phi'_n]|_\mathcal{P} = \alpha|_\mathcal{P}$$

for all sufficiently large n. □

6. The isomorphism with KL

The following is just a variation of the Theorem 3.9 in [Ln10]. An earlier version can be found in [Ln3]. A proof can also be found in [Ln7].

Theorem 6.1. *Let A be a separable unital amenable C^*-algebra and let B be a σ-unital C^*-algebra. Suppose that $h_1, h_2 : A \to B$ are two homomorphisms such that*

$$[h_1] = [h_2] \text{ in } KK(A, B).$$

Suppose that $h_0 : A \to B$ is a full monomorphism such that $[h_0(1_A)] = [h_1(1_A)] = [h_2(1_A)]$. Then, for any $\varepsilon > 0$ and finite subset $\mathcal{F} \subset A$, there is an integer n and a unitary $w \in U(M_{n+1}(B))$ such that

$$\|w^*\mathrm{diag}(h_1(a), h_0(a), \cdots, h_0(a))w - \mathrm{diag}(h_2(a), h_0(a), \cdots, h_0(a))\| < \varepsilon$$

for all $a \in \mathcal{F}$.

Proof. Theorem 3.9 in [Ln10] assumes that B is unital and h_0, h_1 and h_2 are all unital. We will reduce the case here to the unital case. Since h_0 is full, $e = h_0(1_A)$ is a full projection in B. Then there is an integer k such that $h_1(1_A)$ and $h_2(1_A)$ are equivalent to some projections in $EM_k(B)E$, where $E = \mathrm{diag}(e, .., e)$, and where e repeats k times. Without loss of generality, we may assume that $h_1(1_A)$ and $h_2(1_A)$ are in $EM_k(B)E$. Since $[h_1(1_A)] = [h_2(1_A)]$ in $K_0(B)$, there exists an integer m such that $\mathrm{diag}(h_1(1_A), E, ..., E)$ is unitarily equivalent to $\mathrm{diag}(h_2(1_A), E, ..., E)$ in $M_{2mk}(B)$. Set $H_1 = \mathrm{diag}(h_1, h_0, ..., h_0)$ and $H_2 = \mathrm{diag}(h_2, h_0, ..., h_0)$, where h_0 repeats mk times. By replacing H_1 by $\mathrm{ad}\, U \circ H_1$ for some unitaries $U \in M_{2mk+2}(eBe)$ we may assume that

$$\mathrm{diag}(h_1(1_A), h_0(1_A), ..., h_0(1_A)) = \mathrm{diag}(h_2(1_A), h_0(1_A), ..., h_0(1_A)).$$

Set

$$P = \mathrm{diag}(h_1(1_A), h_0(1_A), ..., h_0(1_A)).$$

Let $D = PM_{mk+1}(B)P$, $\Phi = \mathrm{diag}(h_1, h_0, ..., h_0)$, $\Psi = \mathrm{diag}(h_2, h_0, ..., h_0)$ and $H_0 = \mathrm{diag}(h_0, ..., h_0)$, where h_0 repeats mk times in Φ and Ψ and h_0 repeats $mk + 1$ times in H_0. Then this theorem follows from the unital version in [Ln10] (Theorem 3.9). □

Theorem 6.2. *Let A be a unital separable amenable C^*-algebra and let B be a separable unital C^*-algebra. Suppose $\tilde{\Gamma} : KL(A, q_\infty(B_\infty)) \to \tilde{\Gamma}(KL(A, q_\infty(B \otimes \mathcal{O}_\infty)))$ is injective. Then $\beta_B : E_A(B) \to Hom_\Lambda(\underline{K}(A), \underline{K}(B \otimes \mathcal{O}_\infty))$ is also injective.*

Proof. By 5.18, it suffices to show that β_B is injective. Let

$$\alpha \in \tilde{\Gamma}(KL(A, B \otimes \mathcal{O}_\infty)) \subset Hom_\Lambda(\underline{K}(A), \underline{K}(B \otimes \mathcal{O}_\infty))$$

and $\langle\phi\rangle$, $\langle\psi\rangle \in E_A(B)$ such that $\beta_B(\langle\phi\rangle) = \beta_B(\langle\psi\rangle)$. Suppose that $\langle\phi\rangle$ and $\langle\psi\rangle$ are represented by $\phi = \{\phi_n\}$ and $\psi = \{\psi_n\}$, respectively. Since B is unital, we may write $B^\# = B \otimes \mathcal{O}_\infty \otimes \mathcal{K} \oplus \mathcal{O}_\infty \otimes \mathcal{K}$. We may write $\phi_n = \phi_n' \oplus \phi_n''$ and $\psi_n = \psi_n' \oplus \phi_n''$, where $\phi_n', \psi_n' : A \to B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ and $\phi_n'', \psi_n'' : A \to \mathcal{O}_\infty \otimes \mathcal{K}$. Moreover since $\{\pi^\# \circ \phi_n\} \oplus j$ and $\{\pi^\# \circ \psi_n\} \oplus j$ are approximately unitarily equivalent to j, we have that, for any finite subset $\mathcal{P} \subset \mathbf{P}(A)$, there exists $N > 0$ such that

$$[\phi_n']|_{\mathcal{P}} = [\psi_n']|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$$

for all large $n \geq N$. It follows from 2.9 in [GL1] (see also Remark 2.1 in [GL2]) that

$$K_i(l^\infty(B \otimes \mathcal{O}_\infty \otimes \mathcal{K})) = \prod_n K_i(B \otimes \mathcal{O}_\infty),$$

$$K_i(l^\infty(B \otimes \mathcal{O}_\infty \otimes \mathcal{K}), \mathbb{Z}/k\mathbb{Z}) = \prod_n K_i(B \otimes \mathcal{O}_\infty, \mathbb{Z}/k\mathbb{Z}),$$

$$K_i(q_\infty(B \otimes \mathcal{O}_\infty \otimes \mathcal{K})) = \prod_n K_i(B \otimes \mathcal{O}_\infty)/\oplus_n K_i(B \otimes \mathcal{O}_\infty) \quad \text{and}$$

$$K_i(q_\infty(B \otimes \mathcal{O}_\infty \otimes \mathcal{K}, \mathbb{Z}/k\mathbb{Z})) = \prod_n K_i(B \otimes \mathcal{O}_\infty, \mathbb{Z}/k\mathbb{Z})/\oplus_n K_i(B \otimes \mathcal{O}_\infty, \mathbb{Z}/k\mathbb{Z}).$$

Define $\Phi, \Psi : A \to \prod_{n \geq N} B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ by $\Phi_N(a) = \{\phi_n(a)\}_{n \geq N}$ and $\Psi_N(a) = \{\psi_n(a)\}$ for $a \in A$, respectively. We then have

$$[\Psi]|_{\mathcal{P}} = [\Phi]|_{\mathcal{P}}.$$

Let $h_1 = \pi' \circ \Phi$ and $h_2 = \pi' \circ \Psi$, where $\pi' : l^\infty(B \otimes \mathcal{O}_\infty \otimes \mathcal{K}) \to q_\infty(B \otimes \mathcal{O}_\infty \otimes \mathcal{K})$. Regarding $[h_1]$ and $[h_2]$ as elements in $KL(A, q_\infty(B \otimes \mathcal{O}_\infty \otimes \mathcal{K}))$, we have

$$\tilde{\Gamma}_1([h_1]) = \tilde{\Gamma}_1([h_2]).$$

By the assumption that $\tilde{\Gamma}_1$ is injective, we obtain $[h_1] = [h_2]$ in $KL(A, q_\infty(B_\infty))$. Let $J : A \to \mathcal{O}_2 \to l^\infty(\mathcal{O}_\infty \otimes \mathcal{K})$ be defined by $J(a) = \{j(a), j(a), ...\}$. Put $h_3 = h_1 \oplus (\pi' \circ J)$. Note that h_3 is full. It follows from 6.1 that there is a sequence of integers $\{m(n)\}$ and a sequence of unitaries $w_n \in \widetilde{q_\infty(B_\infty)}$ such that

$$\|\mathrm{ad}\, w_n \circ (h_1(a) \oplus d_{m(n)} \circ h_3(a)) - h_2(a) \oplus d_{m(n)} \circ h_3(a)\| \to 0$$

as $n \to \infty$ for all $a \in A$. Let $\{\bar{\phi}'_n\}$ be the sequence of the maps given by 4.5 corresponding to $\{\phi_n\}$. Define $\bar{\Phi} : A \to l^\infty(B_\infty)$ by $\bar{\Phi}(a) = \{\bar{\phi}_n(a)\}$ for $a \in A$. Set $h_4 = \pi' \circ \bar{\Phi}$. Then

$$\|\mathrm{ad}\, w'_n \circ (h_1(a) \oplus d_{m(n)} \circ h_3 \oplus h_4(a)) - h_2(a) \oplus d_{m(n)} \circ h_3 \oplus h_4(a)\| \to 0$$

as $n \to \infty$ for all $a \in A$, where w'_n is a unitary in $U(\widetilde{q_\infty(B_\infty)})$. However, by the choice of $\bar{\phi}'_n$, we have

$$\|\mathrm{ad}\, z_n \circ (h_1(a) \oplus d_{m(n)} \circ \pi' \circ J(a)) - h_2(a) \oplus d_{m(n)} \circ \pi' \circ J(a)\| \to 0$$

as $n \to \infty$ for all $a \in A$, where z_n is a unitary in $U(\widetilde{q_\infty(B_\infty)})$.

Since $\{d_{m(n)} \circ \pi' \circ J\}$ is approximately unitarily equivalent to $\pi' \circ J$, we have

$$\lim_{n \to \infty} \|\mathrm{ad}\, z'_n \circ (h_1 \oplus \pi' \circ J)(a) - h_2(a) \oplus \pi' \circ J(a)\| = 0$$

for all $a \in A$, where z'_n is another sequence of unitaries. There is a unitary $U_{n,k} = \{v_k^{(n)}\}_{k \geq 1} \in \widetilde{l^\infty(B_\infty)}$ such that $\pi(U_{n,k}) = z'_n$, $n = 1, 2, ...$ Let $u_n = v_n^{(n)}$. Then we have

$$\|\mathrm{ad}\, u_n \circ (\phi'_n(a) \oplus j(a)) - \psi'_n(a) \oplus j(a)\| \to 0$$

as $n \to \infty$ for all $a \in A$. From here we conclude that $\langle \phi \rangle = \langle \psi \rangle$. □

Theorem 6.3. *Let A be unital separable amenable C^*-algebra. Then for any $\varepsilon > 0$ and any finite subset $\mathcal{F} \subset A$, there exists $\delta > 0$, a finite subset $\mathcal{G}$ and a finite subset*

$\mathcal{P} \subset \mathbf{P}(A)$ satisfying the following: if B is a unital C^*-algebra and $\phi, \psi : A \to B_\infty$ are two $\mathcal{G}$-δ-multiplicative contractive completely positive linear maps with

$$[\psi]|_{\mathcal{P}} = [\phi]|_{\mathcal{P}},$$

then there exists a unitary $u \in \widetilde{B_\infty}$ such that

$$\mathrm{ad}\, u \circ (\phi \oplus j) \approx_\varepsilon \psi \oplus j \ \text{ on } \mathcal{F}$$

(here we use the fact that $M_2(B_\infty) \cong B_\infty$).

Proof. Suppose the theorem is false. Let $\{\delta_n\}$ be a decreasing sequence of positive numbers such that $\lim_{n\to\infty} \varepsilon_n = 0$, let $\{\mathcal{F}_n\}$ be an increasing sequence of finite subsets of A such that $\bigcup_{n=1}^\infty \mathcal{F}_n$ is dense in A and let $\{\mathcal{P}_n\}$ be an increasing sequence of finite subsets such that $\bigcup_{n=1}^\infty \mathcal{P}_n = \mathbf{P}(A)$. Then there exists $\varepsilon_0 > 0$, a finite subset $\mathcal{F}_0 \subset A$, a sequence of unital C^*-algebras and two sequences of approximately multiplicative contractive completely positive linear maps ψ_n, $\phi_n : A \to B_\infty^{(n)} = B_n \otimes \mathcal{O}_\infty \otimes \mathcal{K}$ such that

$$[\psi_n]|_{\mathcal{P}_n} = [\phi_n]|_{\mathcal{P}_n}, \quad n = 1, 2, ...,$$

and

$$\sup_n \{\max\{\|\mathrm{ad}\, u_n \circ (\psi_n(a) \oplus j(a)) - (\phi_n(a) \oplus j(a))\| : a \in \mathcal{F}_0\}\} \geq \varepsilon_0$$

for all unitaries $u_n \in U(\widetilde{B_\infty^{(n)}})$.

Define $\Psi : A \to l^\infty(\{B_\infty^{(n)}\})$ and $\Phi : A \to l^\infty(\{B_n\})$ by $\Psi(a) = \{\psi_n(a)\}$ and $\Phi(a) = \{\phi_n(a)\}$ for $a \in A$, respectively. Put $h_1 = \pi' \circ \Psi$ and $h_2 = \pi' \circ \Phi$, where $\pi' : l^\infty(\{B_\infty^{(n)}\}) \to q_\infty(\{B_\infty^{(n)}\})$ is the quotient map. It follows from the proof of 6.2 that

$$\tilde{\Gamma}_1([h_1]) = \tilde{\Gamma}_1([h_2]).$$

The same proof of 6.2 shows that there exists an integer m and a unitary in $w \in M_m(q_\infty(\{B_\infty^{(n)}\}))$ such that

$$\|\mathrm{ad}\, w \circ (h_1(a) \oplus \pi' \circ J(a)) - (h_2(a) \oplus \pi' \circ J(a))\| < \varepsilon_0/2$$

for all $a \in \mathcal{F}_0$. There exists a unitary $U = \{v_n\} \in U(q_\infty(\{B_\infty^{(n)}\}))$ such that $\pi'(U) = w$. Therefore there exists an integer N such that for all $n \geq N$,

$$\|\mathrm{ad}\, v_n \circ (\psi_n(a) \oplus j(a)) - (\phi(a) \oplus j(a))\| < \varepsilon_0$$

for all $a \in \mathcal{F}_0$. This gives a contradiction. □

Lemma 6.4. *With the same hypothesis, elements in $E_A(B)$ can be represented by homomorphisms from A to $B \otimes \mathcal{O}_\infty \otimes \mathcal{K}$.*

Proof. Let $\langle \phi \rangle \in E_A(B)$ be represented by $\{\phi_n\}$ and $\xi = \beta_B(\langle \phi \rangle)$. Fix $\varepsilon > 0$ and a finite subset $\mathcal{F}$. Let δ, $\mathcal{G}$ and $\mathcal{P}$ be as required in 6.3. There exists $N > 0$ such that ϕ_n are $\mathcal{G}$-δ-multiplicative and

$$[\phi_n]|_{\mathcal{P}} = \xi|_{\mathcal{P}}$$

for all $n \geq N$. By 5.19, we may assume that ϕ_n maps into B_∞. It follows from 6.3 that there exists, for each n, a unitary $u_{n,k} \in \widetilde{B_\infty}$ such that

$$\|\mathrm{ad}\, u_{n,k} \circ (\phi_n \oplus j)(a) - (\phi_{n+k} \oplus j)(a)\| < \varepsilon$$

for all $a \in \mathcal{F}$ and $n \geq N$. By applying 4.7 we obtain a subsequence $\{m(k)\}$ and a sequence of unitaries $w_k \in \widetilde{B_\infty}$ such that $h(a) = \lim_{n\to\infty} w_k^*(\phi_{m(k)} \oplus j)(a) w_k$

converges for each $a \in A$ and $h : A \to B^{\#}$ is a homomorphism. Furthermore, $[h] = \xi$. By applying 6.3 again, we obtain that $\langle \phi \rangle = \langle h \rangle$. □

Theorem 6.5. *Let A be a unital separable amenable C^*-algebra which satisfies the AUCT and B be a unital separable C^*-algebra. Then*

$$\beta_B : E_A(B) \to KL(A, B \otimes \mathcal{O}_\infty \otimes \mathcal{K})$$

is an isomorphism.

Proof. If A satisfies the AUCT, the map $\tilde{\Gamma}_1$ in 6.2 is an isomorphism. Therefore, this theorem follows immediately from 6.2 and 5.18. □

Theorem 6.6. *Let A be a unital separable amenable C^*-algebra which satisfies the AUCT and B be a unital separable amenable purely infinite simple C^*-algebra. Then $\beta_B : E_A(B) \to KL(A, B)$ is an isomorphism.*

Proof. It is proved in [KP] that, under the assumption in the theorem, $B_\infty \cong B \otimes \mathcal{K}$. Thus this theorem follows from 6.5. □

The following is the one of the main results of this paper.

Theorem 6.7. *Let A be a unital separable amenable C^*-algebra satisfying the AUCT and let B be a unital separable amenable purely infinite simple C^*-algebra. Let $h_i : A \to B$ be two monomorphisms. Then there exists a sequence of isometries $u_n \in U(B)$ such that*

$$\lim_{n\to\infty} \|\mathrm{ad}\, u_n \circ h_1(a) - h_2(a)\| = 0$$

for all $a \in A$ if and only if

$$[h_1] = [h_2] \quad \text{in } KL(A, B).$$

Moreover, for any $x \in KL(A, B)$, there exists a monomorphism $h : A \to B$ such that $[h] = x$.

Proof. The last part of the theorem follows from 6.4 and 6.6.

It was proved by Rørdam (5.4 in [Ro2]) that the "only if" part holds. We will present the proof for the other direction. Let $\varepsilon > 0$ and $\mathcal{F}$ be a finite subset of A. It follows from 6.6 that there exists a unitary $u \in M_2(B)$ such that

$$\|\mathrm{ad}\, u \circ (h_1 \oplus j)(a) - (h_2 \oplus j)(a)\| < \varepsilon/4$$

for all $a \in \mathcal{F}$. It follows from 2.8 in [KP] and 4.11 in [P3] (see also (iii) in 8.2.5 in [Ro4]) that there exists a homomorphism $\phi : B \to B$ which factors through $\mathcal{O}_2$ and an isometry $v \in M_2(B)$ with $v^*v = 1_{M_2(B)}$ and $vv^* = 1_B$ such that

$$\|\mathrm{ad}\, v \circ (\mathrm{id}_B \oplus \phi)(b) - \mathrm{id}_B(b)\| < \varepsilon/8$$

for all $b \in (h_1 \oplus j)(\mathcal{F}) \cup (h_2 \oplus j)(\mathcal{F})$. Since ϕ factors through $\mathcal{O}_2$, it follows from 3.6 in [Ro1] that there are unitaries z_1, $z_2 \in B$ such that

$$\|\mathrm{ad}\, z_1 \circ \phi \circ h_1(a) - j(a)\| < \varepsilon/8$$

and

$$\|\mathrm{ad}\, z_2 \circ \phi \circ h_2(a) - j(a)\| < \varepsilon/8$$

for all $a \in \mathcal{F}$. Therefore we have

$$h_1 \approx_{\varepsilon/8} (\mathrm{id}_B \oplus \phi) \circ h_1 \approx_{\varepsilon/8} h_1 \oplus j \approx_{\varepsilon/4} h_2 \oplus j \approx_{\varepsilon/8} (\mathrm{id}_B \oplus \phi) \circ h_2 \approx_{\varepsilon/8} h_2$$

on $\mathcal{F}$. In other words, $h_1 \approx_\varepsilon h_2$ on $\mathcal{F}$. □

7. Relative weakly semiprojectivity for separable amenable C^*-algebras

Lemma 7.1. *Let G_n be a sequence of abelian groups and let G_0 be a finitely generated abelian group. Let $\lambda : G_0 \to \prod_n G_n/\oplus_n G_n$ be a homomorphism. Then there is a homomorphism $\sigma : G_0 \to \prod_n G_n$ such that $\pi\circ\sigma = \lambda$, where $\pi : \prod_n G_n \to \prod_n G_n/\oplus_n G_n$ is the quotient map.*

Proof. Write $G_0 = G_0' \oplus G_0''$, where G_0' is a free group and G_0' is a finite abelian group. It suffices to prove the lemma for the case that G_0 is finite. Suppose that $G_0 = G_{01}\oplus\cdots\oplus G_{0k}$, where each $G_{0,i}$ is cyclic. Let $s_i \in G_{0i}$ be the generator and let $g(i,n)\in G_n$ such that $\pi(\{g(i,n)\}) = \lambda(s_i)$, $i = 1,2,...,k$. Suppose that r_i is the order of $\lambda(s_i)$. Then, for each i, there is $n_i > 0$ such that

$$r_i g(i,n) = 0 \ \text{ for all } \ n > n_i.$$

Choose $N = \max\{n_i : i = 1,2,...,k\}$. Define $\sigma : G_0 \to \prod_n G_n$ by $\sigma(s_i) = (0,...,0,g(i,N+1),g(i,N+2),...)$, $i = 1,2,...,k$. Clearly σ gives a homomorphism and $\pi\circ\sigma = \lambda$. □

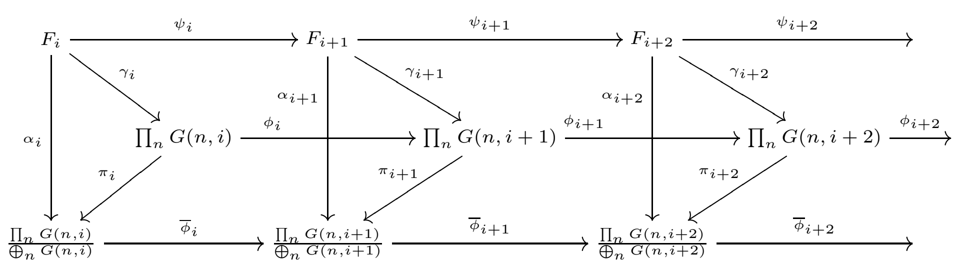
Lemma 7.2. *Let $\{G(n,1)\},\{G(n,2)\},...,\{G(n,k)\}$ be k sequences of abelian groups and let $F_1,...,F_k$ be k finitely generated abelian groups. Let $\phi_i : \prod_n G(n,i) \to \prod_n G(n,i+1)$ and $\psi_i : F_i \to F_{i+1}$ be homomorphisms. Denote by*

$$\bar\phi_i : \prod_n G(n,i)/\bigoplus_n G(n,i) \to \prod_n G(n,i+1)/\bigoplus_n G(n,i+1)$$

the induced homomorphism by ϕ_i, $i = 1,2,...,k$. Suppose that there is a homomorphism $\alpha_i : F_i \to \prod_n G(n,i)/\oplus_n G(n,i)$ such that the diagram

$$\begin{array}{ccccc}
\to & F_i & \xrightarrow{\psi_i} & F_{i+1} & \\
 & \downarrow_{\alpha_i} & & \downarrow_{\alpha_{i+1}} & \\
\to & \prod_n G(n,i)/\bigoplus_n G(n,i) & \xrightarrow{\bar\phi_i} & \prod_n G(n,i+1)/\bigoplus_n G(n,i+1) & \\
 & \xrightarrow{\psi_{i+1}} & F_{i+2} & \xrightarrow{\psi_{i+2}} & \\
 & & \downarrow_{\alpha_{i+2}} & & \\
 & \xrightarrow{\bar\phi_{i+1}} & \prod_n G(n,i+2)/\bigoplus_n G(n,i+2) & \xrightarrow{\bar\phi_{i+2}} &
\end{array}$$

commutes for each $i = 1,2,...,k$. Then there is a homomorphism $\gamma_i : F_i \to \prod_n G(n,i)$ such that the following diagram commutes for every i:



where $\pi_i : \prod_n G(n,i) \to \prod_n G(n,i)/\bigoplus_n G(n,i)$ is the quotient map.

Moreover, there exists $N \geq 1$ satisfying the following: if $\bar\gamma_i : F_i \to \prod_n G(n,i)$ is another homomorphism such that $\pi_i\circ\bar\gamma_i = \pi_i\circ\gamma_i$, then

$$p_{n,i}\circ\gamma_i = p_{n,i}\circ\bar\gamma_i$$

for $n \geq N$ and $i = 1, 2, ..., k$, where $p_{n,i} : \prod_n G(n,i) \to G(n,i)$ is the projection. Consequently, if $F_k = F_1$ and $\alpha_1 = \alpha_k$, then we can choose $\gamma_k = \gamma_1$.

Proof. It follows from 7.1 that there is $\sigma_i : F_i \to \prod_n G(n,i)$ such that $\pi_i \circ \sigma_i = \alpha_i$. Let $\{s(i,j) : j = 1, 2, ..., m(i)\}$ be the cyclic generators of F_i. Denote by $p_{n,i} : \prod_n G(n,i) \to G(n,i)$ the projection, $i = 1, 2, ..., k$. For each i, since $\bar{\phi}_i \circ \alpha_i = \alpha_{i+1} \circ \psi_i$, there is an integer $N(i,j) > 0$ such that

$$p_{n,i} \circ \phi_i \circ \sigma_i(s(i,j)) = p_{n,i+1} \circ \sigma_{i+1} \circ \psi_i(s(i,j))$$

for all $n \geq N(i,j)$, $j = 1, 2, ..., m(i)$ and $i = 1, 2, ..., k$. Choose $N_1 = \max\{N(i,j) : 1 \leq j \leq m(i), 1 \leq i \leq k\}$. Define $\gamma_i : F_i \to \prod_n G(n,i)$ by

$$\gamma_i(s(i,j)) = (0, ..., 0, p_{N_1+1,i} \circ \sigma_i(s(i,j)), p_{N+2,i} \circ \sigma_i(s(i,j)), ..., p_{N_1+m,i} \circ \sigma_i(s(i,j)), ...),$$

$j = 1, 2, ..., m(i), i = 1, 2, ..., k$. Then we have

$$\phi_i \circ \gamma_i = \gamma_{i+1} \circ \psi_i \quad \text{and} \quad \alpha_i = \pi_i \circ \gamma_i$$

for all i. Moreover we have the commutative diagram in the statement of the lemma for every i.

If $\bar{\gamma}_i : F_i \to \prod_n G(n,i)$ is another homomorphism such that $\pi_i \circ \gamma_i = \pi_i \circ \bar{\gamma}_i$, then, as above, there is $N(i)$ such that $p_{n,i} \circ \gamma_i = p_{n,i} \circ \bar{\gamma}_i$ for all $n \geq N(i)$, $i = 1, 2, ..., k$. Choose $N = \max\{N(i) : i = 1, 2, ..., k\}$.

If $F_k = F_1$ and $\alpha_k = \alpha_1$, as above, by choosing possibly even larger N_1, we may choose $\gamma_k = \gamma_1$. □

The following two lemmas are taken from [DR] (1.3 in [DR]).

Lemma 7.3. (i) *Let A be a separable C^*-subalgebra in an amenable C^*-algebra B. There exists a separable amenable C^*-subalgebra C such that $A \subset C \subset B$.*

(ii) *Let A be a separable C^*-algebra in an amenable purely infinite simple C^*-algebra B. Then there is a separable amenable purely infinite simple C^*-algebra $C \subset B$ such that $A \subset C$.*

Lemma 7.4. *Let A be a separable unital amenable C^*-algebra and let $h : A \to q_\infty(\{B_n\})$ be a full homomorphism, where B_n are separable purely infinite simple C^*-algebra. Let $j_n : A \to B_n$ be a monomorphism which factors through $\mathcal{O}_2$ and let $\bar{j} : A \to q_\infty(\{B_n\})$ be defined by $\pi \circ \{j_n\}$, where $\pi : l^\infty(\{B_n\}) \to q_\infty(\{B_n\})$ is the quotient map. Then there exists a partial isometry $u \in M_2(q_\infty(\{B_n\}))$ such that $u^*u = 1_{B_n}$ and $uu^* = 1_{M_2(B_n)}$ and*

$$\mathrm{ad}\, u_n \circ (h \oplus \bar{j}) = h.$$

Proof. Since A is amenable, there exists a contractive completely positive linear map $L : A \to l^\infty(\{B_n\})$ such that $\pi \circ L = h$. Write $L = \{\psi_n\}$, where each $\psi_n : A \to B_n$ is a contractive completely positive linear map. Let $\{\mathcal{F}_n\}$ be an increasing sequence of finite subsets of A such that $\bigcup_{n=1}^\infty \mathcal{F}_n$ is dense in A. There is a sequence of projections $e_n \in B_n$ such that

$$\|\psi_n(1_A) - e_n\| \to 0 \quad \text{as } n \to \infty.$$

Without loss of generality, by replacing ψ_n by $e_n \psi_n e_e$, we may assume that $\psi_n(1_A) \leq e_n$ for all n. By replacing B_n by $e_n B_n e_n$, we may also assume that B_n are unital. It follows from (iii) in [Ro4] (see also 2.4 in [P2]) that there is an embedding

$\phi_n : B_n \to B_n$ which factors through $\mathcal{O}_2$ and a sequence of isometry $\{u_n\}$ in $M_2(B_n)$ such that $u_n^* u_n = 1_{M_2(B_n)}$, $u_n u_n^* = 1_{B_n}$ and

$$\|u_n(\mathrm{id}_{B_n} \oplus \phi_n)(b)u_n^* - \mathrm{id}_{B_n}(b)\| < 1/n$$

for all $b \in \phi_n(\mathcal{F}_n)$. We write $\phi_n = \phi_n^{(1)} \circ \phi_n^{(0)}$, where $\phi_n^{(0)} : B_n \to \mathcal{O}_2$ and $\phi_n^{(1)} : \mathcal{O}_2 \to B_n$ are monomorphisms. Let $\bar{\phi} = \pi \circ \{\phi_n\}$, $\sigma = \bar{\phi} \circ h$, $\bar{\phi}^{(1)} = \pi \circ \{\phi_n^{(1)}\}$ and $\bar{\phi}^{(0)} = \pi \circ \{\phi_n^{(0)}\}$. Since h and $\bar{\phi}^{(0)}$ are full, so is $\bar{\phi}^{(0)} \circ h$. It follows from 3.4 that there is monomorphism $h_0 : A \to l^\infty(\mathcal{O}_2)$ such that $\pi' \circ h_0 = \bar{\phi}^{(0)} \circ h$. Therefore we have

$$\|\phi_n^{(0)} \circ \psi_n(a) - h_n^{(0)}(a)\| \to 0 \quad \text{as } n \to \infty$$

for all $a \in A$, where $h_0(a) = \{h_n^{(0)}(a)\}$. It follows from 5.11 in [Ro4] that there are unitaries $z_n \in B_n$ such that

$$\|\mathrm{ad}\, z_n \circ \phi_n^{(1)} \circ h_n^{(0)}(a) - j_n(a)\| \to 0 \quad \text{as } n \to \infty$$

for all $a \in A$. Combining all these, we obtain a sequence of partial isometries $w_n \in M_2(B_n)$ with $w_n^* w_n = 1_{B_n}$ and $w_n w_n^* = 1_{M_2(B_n)}$ such that

$$\|\mathrm{ad}\, w_n(\psi_n(a) \oplus j_n(a)) - \psi_n(a)\| \to 0 \quad \text{as } n \to \infty$$

for all $a \in A$. Let $w = \{w_n\} \in M_2(l^\infty(\{B_n\}))$ and $u = \pi(w)$. Then

$$\mathrm{ad}\, u \circ (h \oplus \bar{j}) = h.$$

□

Theorem 7.5. *Let A be a unital separable amenable C^*-algebra with finitely generated $K_i(A)$ $(i = 0, 1)$ satisfying the AUCT. Let $\mathbf{B}$ be the class of amenable purely infinite simple C^*-algebras. Then A is apf-weakly semiprojective with respect to $\mathbf{B}$.*

Proof. Let $\{B_n\}$ be a sequence of amenable C^*-algebras in $\mathbf{B}$ and $h : A \to q_\infty(B_n)$ be a full homomorphism. Let $L : A \to l^\infty(B_n)$ be a contractive completely positive linear map such that $\pi \circ L = h$, where $\pi : l^\infty(\{B_n\}) \to q_\infty(\{B_n\})$ is the quotient map. Write $L = \{L_n\}$, where each $L_n : A \to B_n$ is a contractive completely positive linear map. Let A_n be the separable C^*-subalgebra generated by $L_n(A)$. It follows from 7.3 that there is an amenable separable purely infinite simple C^*-algebra C_n such that $L_n(A) \subset C_n \subset B_n$. Thus we may replace B_n by C_n. Therefore we may assume that each B_n is separable. Since $B_n \subset B_n \otimes \mathcal{K}$ and $q_\infty(\{B_n\}) \subset q_\infty(\{B_n \otimes \mathcal{K}\})$, we now consider the case that each B_n is stable.

Since $K_i(A)$ $(i = 0, 1)$ is finitely generated, by 2.11 in [DL2] there is an integer $k_0 > 0$ such that $Hom_\Lambda(\underline{K}(A), \underline{K}(C)) \cong Hom_\Lambda(F_{k_0}\underline{K}(A), F_{k_0}\underline{K}(C))$ for all σ-unital C^*-algebra C, where $F_{k_0}\underline{K}(D) = K_*(A) \oplus \bigoplus_{k \le k_0} K_*(D, \mathbb{Z}/k\mathbb{Z})$. Let $\tilde{\Gamma}([h]) \in Hom_\Lambda(F_{k_0}\underline{K}(A), F_{k_0}\underline{K}(q_\infty(B_n)))$.

Put

$$P_0^{(i)} = \prod_n K_i(B_n), P_k^{(i)} = \prod_n K_i(B_n, \mathbb{Z}/k\mathbb{Z}),$$

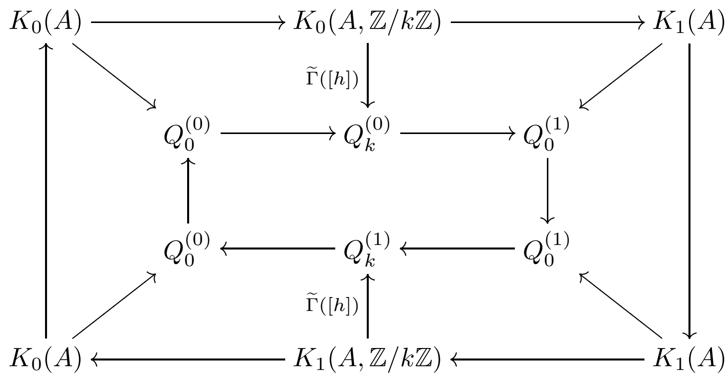
$$Q_0^{(i)} = \prod_n K_i(B_n)/\bigoplus_n K_i(B_n) \quad \text{and} \quad Q_k^{(i)} = \prod_n K_i(B_n, \mathbb{Z}/k\mathbb{Z})/\bigoplus_n K_i(B_n, \mathbb{Z}/k\mathbb{Z}).$$

It follows from 2.9 in [GL1] that

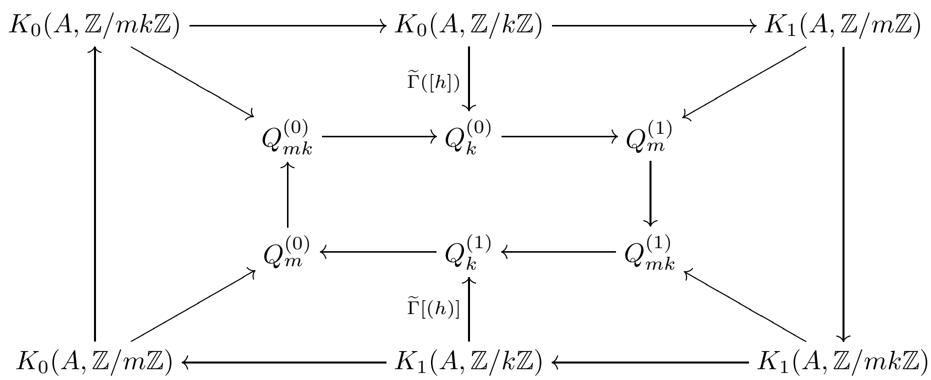
$$K_i(l^\infty(B_n \otimes \mathcal{K})) = P_0^{(i)}, \ \ K_i(l^\infty(B_n \otimes \mathcal{K}), \mathbb{Z}/k\mathbb{Z}) = P_k^{(i)},$$

$$K_i(q_\infty(B_n \otimes \mathcal{K})) = Q_0^{(i)} \quad \text{and} \quad K_i(q_\infty(B_n \otimes \mathcal{K}), \mathbb{Z}/k\mathbb{Z}) = Q_k^{(i)},$$

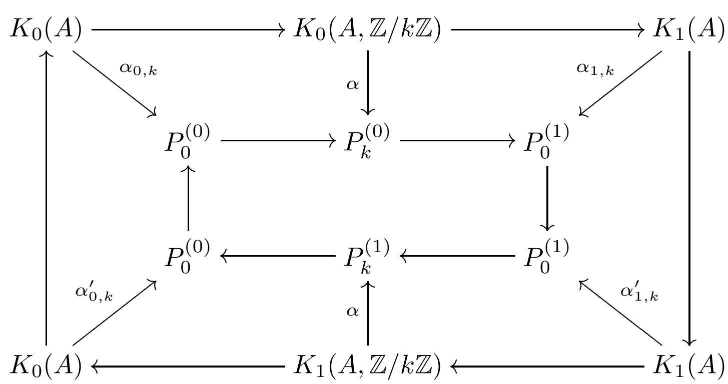
$k = 2, 3, \ldots$. We have the following commutative diagrams:



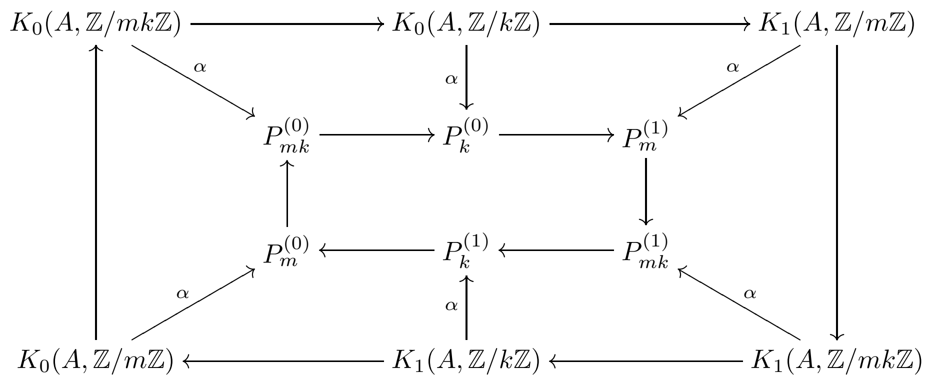
and



It follows from 7.2 that for any $0 \leq k \leq k_0^2$, we obtain homomorphisms $\alpha : K_i(A, \mathbb{Z}/k\mathbb{Z}) \to P_k^{(i)}$, $\alpha_{i,k} : K_i(A) \to P_0^{(i)}$ and $\alpha'_{i,k} : K_i(A) \to P_0^{(i)}$ such that $\pi_* \circ \alpha = \tilde{\Gamma}([h])|_{(A,\mathbb{Z}/k\mathbb{Z})}$, $\pi_* \circ \alpha_{i,k} = h_{*i}$ and $\pi_* \circ \alpha'_{i,k} = h_{*i}$, $i = 0, 1$, and $0 \leq k \leq k_0^2$. Furthermore we have the commutative diagram



for all $k \leq k_0^2$. By the last part of 7.2, there is an integer $N > 0$ such that by replacing $p_{n,i} \circ \alpha, p_{n,i} \circ \alpha_{i,k}$, and $p_{n,i} \circ \alpha'_{i,k}$ by zero maps for $n \leq N$, we may assume that $\alpha_{i,k} = \alpha'_{i,k}$, $0 \le k \le k_0^2$ and $i = 0, 1$. Similarly, we have the commutative diagram



for all $k \le k_0$. It follows from 2.11 in [DL2] that there is

$$\alpha = \{\alpha_n\} \in Hom_\Lambda(F_{k_0}\underline{K}(A), F_{k_0}\underline{K}(l^\infty(B_n))$$

such that $\pi_* \circ \alpha = \Gamma([h])$. Note that each $\alpha_n \in Hom_\Lambda(F_{k_0}\underline{K}(A), F_{k_0}\underline{K}(B_n))$. It follows from 6.4 and 6.5 that there is a homomorphism $h_n : A \to B_n$ such that $[h_n] = \alpha_n$. Fix a sequence of finite subsets $\mathcal{F}_j$ such that $\mathcal{F}_j \subset \mathcal{F}_{j+1}$, $n = 1, 2, ...,$ and $\bigcup_{n=1}^\infty \mathcal{F}_n$ is dense in A, and a decreasing sequence of positive numbers ε_j such that $\lim_{j\to\infty} \varepsilon_j = 0$. For each j, let $\delta_j > 0$, $\mathcal{P}_j \subset \mathbf{P}(A)$ and $\mathcal{G}_j$ be finite subsets associated with ε_j, $\mathcal{F}_j$ and A as required by 6.3. There is $n(j) > 0$ such that $L_n : A \to B_n$ is $\mathcal{G}_n$-δ_n-multiplicative, and $[L_n]|_{\mathcal{P}_j}$ is well defined for all $n \ge n(j)$. Furthermore, we may also assume (with perhaps even larger $n(j)$) that

$$[L_n]|_{\mathcal{P}_j} = (\alpha_n)|_{\mathcal{P}_j}$$

for all $n \ge n(j)$. It follows from 6.3 that there is a unitary $u(j,n) \in \tilde{B}_n$ such that

$$L_n \oplus j_n \approx_{\varepsilon_j} \mathrm{ad}\, u(j,n) \circ (h_n \oplus j_n) \ \text{ on } \ \mathcal{F}_j.$$

In the proof of 6.7, we absorb j by applying 4.1.1 in [P3] (also (iii) in [Ro4]). Here we apply 7.4. As in the proof of 6.7, we have ($z_n \in U(\tilde{B}_n)$)

$$\|\mathrm{ad}\, \mathrm{z_n} \circ (L_n(a) \oplus j_n(a)) - L_n(a)\| \to 0, \quad \text{as } \ n \to \infty,$$

for all $a \in A$. Similarly, there are unitaries $v(j,n) \in \tilde{B}_n$ such that

$$\|\mathrm{ad}\, v(j,n) \circ (h_n \oplus j_n)(a) - h_n(a)\| \to 0, \ \text{ as } \ n \to \infty.$$

Thus (with $n > (n(j))' \ge n(j)$) we have

$$L_n \approx_{2\varepsilon_j} \mathrm{ad}\, w(j,n) \circ h_n \ \text{ on } \ \mathcal{F}_j,$$

where $w(j,n)$ is also a unitary.

Define $w_1 = 1, ..., w_{n(1)-1} = 1, w_{n(j)+i} = u(j, n(j)+i)$, $0 \le i \le n(j+1) - n(j) - 1$ and $\phi_n = \mathrm{ad}\, w_n \circ h_n$. Then, since $\bigcup_{j=1}^\infty \mathcal{F}_j$ is dense in A, we conclude that

$$\lim_{n\to\infty} \|L_n(a) - \phi_n(a)\| = 0 \ \text{ for all } \ a \in A.$$

Finally define $H(a) = \{\phi_n(a)\}$ for $a \in A$. Then $H : A \to l^\infty(\{B_n\})$ is a homomorphism. Moreover, $\pi \circ H = h$. We write $H = \{H_n\}$, where each $H_n : A \to B_n \otimes \mathcal{K}$ is a homomorphism. Put $p_n = H_n(1_A)$. Since $\|p_n - L_n(1_A)\| \to 0$ (as $n \to \infty$) and

$L_n(1_A) \in B_n$, there is a sequence of projections $e_n \in B_n$ such that $\|p_n - e_n\| \to 0$ as $n \to \infty$. We obtain unitaries $v_n \in \widetilde{B_n \otimes \mathcal{K}}$ such that

$$\|v_n - 1\| \to 0 \text{ as } n \to \infty, v_n^* p_n v_n = e_n \quad \text{and} \quad v_n e_n v_n^* = p_n, \ n = 1, 2, ...$$

Put $H_n'(a) = v_n^* H_n(a) v_n$ for $a \in A$, $n = 1, 2, ...$ Then $H_n' : A \to B_n$ is a homomorphism and

$$\|H_n' - H\| \to 0 \text{ as } n \to \infty.$$

Therefore

$$\lim_{n\to\infty} \|L_n(a) - H_n'(a)\| = 0 \text{ for all } a \in A.$$

Put $H'(a) = \{H_n(a)\}$. Then $\pi \circ H' = h$. □

Corollary 7.6. *Let A be as in 7.5. Then A is api-weakly stable.*

Proof. This follows from 7.5 and 2.7 immediately. □

The proof of the following is similar to the last part of the proof of 3.4.

Theorem 7.7. *Let A be a unital separable simple amenable C^*-algebra with finitely generated $K_i(A)$ which satisfies the AUCT. Then A is weakly stable respect to* **B**.

Proof. Let $\{B_n\}$ be a sequence of purely infinite simple amenable C^*-algebra and let $h : A \to q_\infty(\{B_n\})$ be a homomorphism. Let $q = h(1_A)$. Then there exists a projection $Q \in l^\infty(\{B_n\})$ such that $\pi(Q) = p$. Write $Q = \{p_n\}$, where $p_n \in B_n$ is a projection. There exists a subsequence $\{m(k)\}$ such that $p_n \neq 0$ if $n = m(k)$ for some k otherwise $p_n = 0$. Let $P : l^\infty(\{B_n\}) \to l^\infty(\{B_{m(k)}\})$ be the projection and $\bar{P} : q_\infty(\{B_n\}) \to q_\infty(\{B_{m(k)}\})$ be the map induced by P (P maps $c_0(\{B_n\})$ to $C_0(\{B_{m(k)}\})$). Consider $h' = \bar{P} \circ h : A \to q_\infty(\{B_{m(k)}\})$. Since A is simple, for any $a \in A_+ \setminus \{0\}$, there are $x_1, ..., x_m \in A$ such that

$$\sum_{i=1}^m h'(x_i)^* h'(a) h'(x_i) = h'(1_A).$$

For any $b \in q_\infty(\{B_n\}) \setminus \{0\}$, there is a sequence $\{b_n\}$ such that $b_n \in (B_n)_+$, $\|b_n\| \leq \|b\|$ and $\pi(\{b_n\}) = b$. Since B_n is purely infinite and simple, there exists $y_n \in B_n$ such that $\|y_n\| \leq \|b_n\| \leq \|b\|$ such that

$$y_n^* p_n y_n = b_n \quad \text{for all } n.$$

In fact one can choose $y_n = 1_{B_n} z_n b_n^{1/2}$, where z_n is a partial isometry in $M_2(B_n)$ so that $z_n^* p_n z_n \geq 1_{B_n}$. Let $\pi' : l^\infty(\{B_{m(k)}\}) \to q_\infty(\{B_{m(k)}\})$. Set $y = \pi'(\{y_n\})$. Then we have

$$\sum_{i=1}^m y^* h'(x_i)^* h'(a) h'(x_i) y = b.$$

This implies that h' is full. Therefore by 7.5 there exists a homomorphism $\tilde{h}' : A \to l^\infty(\{B_{m(k)}\})$ such that $\pi' \circ \tilde{h}' = h'$. Write $\tilde{h}' = \{h_{m(k)}\}$, where each $h_{m(k)} : A \to B_{m(k)}$ is a homomorphism. Define $\tilde{h} = \{h_n\}$, where $h_n = h_{m(k)}$ if $n = m(k)$; otherwise $h_n = 0$. Then we have $\pi \circ \tilde{h} = h$. □

Corollary 7.8. *For any $\varepsilon > 0$ there is $\delta > 0$ satisfying the following: for any C^*-algebra $A \in \mathbf{B}$, two unitaries $u, v \in A$ and an irrational number θ such that*

$$\|uv - e^{i\theta\pi}vu\| < \delta,$$

there exist two unitaries u_1 and v_1 in A such that

$$u_1v_1 = e^{i\theta\pi}v_1u_1, \ \|u - u_1\| < \varepsilon \quad \text{and} \quad \|v - v_1\| < \varepsilon.$$

Proof. Let A_θ be the irrational rotation C^*-algebra which is generated by two unitaries u and v with the relation $uv = e^{i\theta\pi}vu$. We also know that $K_i(A_\theta) = \mathbb{Z}\oplus\mathbb{Z}$ as abelian groups. Furthermore A_θ is a simple unital separable amenable C^*-algebra. Then the corollary follows from the fact that A_θ is weakly stable with respect to $\mathbf{B}$. □

8. Direct sums of finitely generated abelian groups

Proposition 8.1. *Let G be a countable abelian group. Then the following are equivalent:*

(1) *$G = \bigcup_{n=1}^\infty G_n$, where $G_n \subset G_{n+1}$ and each G_n is finitely generated and for all n, there exists $m(n) \geq n$ such that there is $\psi_n : G \to G_{m(n)}$ satisfying*

$$(\psi_n)|_{G_n} = \mathrm{id}_{G_n}.$$

(2) *For any sequence of abelian groups F_n, and any homomorphism $h : G \to \prod_{n=1}^\infty F_n/\bigoplus_{n=1}^\infty F_n$, there exists $\psi : G \to \prod_{n=1}^\infty F_n$ such that $\kappa\circ\psi = h$, where $\kappa : \prod_{n=1}^\infty F_n \to \prod_{n=1}^\infty F_n/\bigoplus_{n=1}^\infty F_n$ is the quotient map.*

Proof. (1) $\Rightarrow$ (2). Let F_n be a sequence of abelian groups F_n and let $h : G \to \prod_{n=1}^\infty F_n/\bigoplus_{n=1}^\infty F_n$ be a homomorphism.

For each j, it follows from 7.1 that there exists a homomorphism $\Phi_j : G_j \to \prod_{n=1}^\infty F_n$ such that $\kappa\circ\Phi_j = h|_{G_j}$. We write $\Phi_j = \{\phi_n^{(j)}\}$, where $\phi_n^{(j)} : G_j \to F_n$ is a homomorphism. Let $\psi_n : G \to G_{m(n)}$ be given by (1).

Define $n(1) = m(1)$. Since $\kappa\circ\Phi_n(g) = \kappa\circ\Phi_1(g)$ for all $g \in G_1$, there exists $n(2) > \max\{m(2), n(1)\}$ such that

$$\phi_{n'}^{(m(2))}|_{G_1} = \phi_{n'}^{(m(1))} \quad \text{for all } n' \geq n(2).$$

If $n(k)$ is defined, find $n(k+1) > \max\{m(k+1), n(k)\}$ such that

$$\phi_{n'}^{(m(k+1))}|_{G_k} = \phi_{n'}^{(m(k))} \quad \text{for all } n' \geq n(k+1).$$

Define $s(1) = m(1), s(2) = m(1), ..., s(n(2)) = m(2), ..., s(k) = m(j)$, if $n(j) \leq s < n(j+1)$, $j = 1, 2, ...$ Define $\psi(g) = \{\psi_k^{(s(k))}\circ\phi_{s(k)}(g)\}$ for $g \in G$.

To verify $\kappa\circ\psi = h$, we let $g \in G_j$. Then for all $k \geq n(j)$,

$$\psi_k^{(s(k))}\circ\psi_{s(k)}(g) = \psi_k^{(s(k))}(g) = \psi_k^{(n(j))}(g)$$

for all $g \in G_j$. This implies that

$$\kappa\circ\psi(g) = h(g)$$

if $g \in G_j$ for all j.

(2) $\Rightarrow$ (1). Since G is countable, let $\{g_n\}$ be a set of generators and let G_n be generated by $g_1, ..., g_n$. We write $G = \bigcup_{n=1}^\infty G_n$, where each G_n is finitely generated and $G_n \subset G_{n+1}$. Denote by $\imath_n$ the homomorphism from G_n to $\prod_{n=1}^\infty G_n$, and by $\imath_n(g) = (0, ..., 0, g, g, ..., g, ...)$ (there are $n-1$ zero's) for $g \in G_n$. This gives a homomorphism $h : G \to \prod_{n=1}^\infty G_n/\bigoplus_{n=1}^\infty G_n$.

Condition (2) gives a homomorphism $\psi : G \to \prod_{n=1}^{\infty} G_n$ such that $\kappa \circ \psi = h$. Write $\psi = \{\psi_n\}$, where $\psi_n : G \to G_n$ is a homomorphism. For each j, there is $m(j)$ such that

$$\psi_m(g) = g \quad \text{for all } m \geq m(j).$$

This implies (1). □

Corollary 8.2. *Let $F_1, F_2, ..., F_k$ be k countable abelian groups which satisfy one of the conditions in 8.1. Then the conclusion of 7.2 holds.*

Proposition 8.3. *Let G be a countable abelian group. Then G satisfies one of the conditions in 8.1 if and only if it is a direct sum of finitely generated abelian groups.*

Proof. Let $G = \bigoplus_{n=1}^{\infty} G_n$, where each G_n is a finitely generated abelian group. It is easy to see that G satisfies condition (1) in 8.1.

To see the converse, we apply (1) in 8.1. Write $G = \bigcup_{n=1}^{\infty} G_n$ as in (1) in 8.1. Define a homomorphism $\Psi : G \to \bigoplus_{n=1}^{\infty} G_{m(n)}$ by $\Psi(g) = \bigoplus_{n=1}^{\infty} \psi_n(g)$ for $g \in G$. It is a homomorphism. For any $g \in G_n$, $\psi_n(g) = g$ (by (1) in 8.1) which implies that Ψ is injective. Thus G is isomorphic to a subgroup of the direct sum $\bigoplus_{n=1}^{\infty} G_{m(n)}$. It follows from 18.1 in [F] that G is also a direct sum of cyclic groups. Hence G is a direct sum of finitely generated abelian groups. □

Theorem 8.4. *Let A be a unital separable amenable C^*-algebra such that $K_0(A)$ or $K_1(A)$ is not a direct sum of finitely generated groups. Suppose that $A = \lim_{n\to\infty}(A_n, f_n)$, where each A_n is a unital separable amenable C^*-algebra with finitely generated $K_i(A_n)$ $(i = 0, 1)$. Then*

(1) *A is not weakly semiprojective with respect to* **B**,
(2) *A is not weakly stable with respect to* **B**,
(3) *A is not apf-weakly semiprojective with respect to* **B** *and*
(4) *A is not api-weakly stable with respect to* **B**.

Proof. It follows from 2.7 and 2.4 that it suffices to show that A is not api-weakly stable with respect to **B**.

Denote by $f_{n,m}$ $(m > n)$ and $f_{n,\infty}$ the homomorphisms from A_n to A_m and from A_n to A induced by the inductive limit, respectively. Let $G_{n,i} = (f_{n,\infty})_*(K_i(A_n))$. Then each $G_{n,i}$ is finitely generated. It follows from [Ro2] that there is a separable amenable purely infinite simple C^*-algebra B_n such that $K_i(B_n) = G_{n,i}$. It follows from 6.4 and 6.5 that there is a monomorphism $h_n : A_n \to B_n$ such that $(h_n)_{*i} = (f_{n,\infty})_{*i}$, $i = 0, 1$. Let $\{\mathcal{F}_n\}$ be an increasing sequence of finite subsets of A such that $\mathcal{F}_n \subset A_n$ and $\bigcup_n \mathcal{F}_n$ is dense in A. Since both A_n and B_n are amenable, there exists a contractive completely positive linear map $\phi_n : A \to B_n$ such that

$$\|\phi_n(a) - h_n(a)\| < 1/2^n$$

for all $a \in \mathcal{F}_n$. Define $L : A \to \prod_{n=1}^{\infty} l^{\infty}(\{B_n\})$ by $L(a) = \{\phi_n(a)\}$ for $a \in A$. It is easy to see that $h = \pi \circ L$ is a homomorphism from A to $q_{\infty}(\{B_n\})$, where $\pi : l^{\infty}(\{B_n\}) \to q_{\infty}(\{B_n\})$ is the quotient map.

For each n, there is $k(n)$ such that $[\phi_m \circ f_{n,m}]|_{K_i(A_n)}$ is well defined and $[\phi_m \circ f_{n,m}] = (f_{n,\infty})_{*i}$ for $m \geq k(n)$ and $i = 0, 1$.

We note that for each j, there is $m(j)'$ such that

$$\|h_n(a)\| \geq (1/2)\|a\| \quad \text{for all } a \in \mathcal{F}_j$$

if $n \geq m(j)'$. If A were api-weakly stable, then there would be an integer $m(j) \geq m(j)'$ and a homomorphism $g_n^{(j)} : A \to A_n$ such that

$$g_n^{(j)} \approx_{1/j} h_n \text{ on } \mathcal{F}_j$$

for all $a \in \mathcal{F}_j$. Define

$$\psi_n(a) = g_{m(j)}^{(j)} \text{ if } m(j) \leq n < m(j+1),$$

$j = 1, 2, ...,$ and define $\Psi : A \to l^\infty(\{A_n\})$ by $\Psi(a) = \{\psi_n(a)\}$ for all $a \in A$. Then

$$\psi_n \approx_{1/j} h_n \text{ on } \mathcal{F}_j$$

for all $n \geq m(j)$ and $j = 1, 2, ...$ Thus

$$\|\psi_n(a) - h_n(a)\| \to 0 \text{ as } n \to \infty$$

for all $a \in A$. Hence, for any fix n, there is $K(n) \geq k(n)$ such that

$$(\psi_m \circ f_{n,\infty})_{*i} = (f_{n,\infty})_{*i}$$

for all $m \geq K(n)$ and $i, = 0, 1$, where we also identify $K_*(B_n)$ with $G_{n,i}$. This implies that

$$(\psi_m)_{*i}|_{G_{n,i}} = \mathrm{id}_{G_{n,i}}.$$

We also have $K_i(A) = \bigcup_{n=1}^\infty G_{n,i}$ and $G_{n,i}$ is finitely generated. Now $(\psi_m)_{*i}$ gives a homomorphism from $K_i(A)$ to $G_{m,i}$ such that $(\psi_m)_{*i}|_{G_{n,i}} = \mathrm{id}_{G_{n,i}}$. Therefore $K_i(A)$ is a direct sum of finitely generated abelian groups (see 8.3). We reach a contradiction. □

Theorem 8.5. *Let A be a separable unital simple amenable C^*-algebra satisfying the AUCT. Suppose that $K_i(A)$ is a direct sum of finitely generated groups with finite torsion for $i = 0, 1$. Then A is weakly semiprojective with respect to* $\mathbf{B}$.

Proof. Since torsion part of $K_*(A)$ is finite, as in the proof of 7.5, we note that (by 2.11 in [DL1]) there is $k_0 > 0$ such that

$$Hom_\Lambda(\underline{K}(A), \underline{K}(C)) \cong Hom_\Lambda(F_{k_0}\underline{K}(A), F_{k_0}\underline{K}(C))$$

for all σ-unital C^*-algebra C. In the proof of 7.5, we also use the fact that $K_i(A)$ is finitely generated so that we can apply 7.2. Here we apply 8.2. The rest of the proof is exactly the same as that of 7.5. □

Corollary 8.6. *Let A be a separable simple unital AF-algebra. Then A is weakly semiprojective with respect to* $\mathbf{B}$ *if and only if $K_0(A)$ is free.*

Proof. Note that $K_1(A) = 0$ and $K_0(A)$ is torsion free. So the corollary follows from 8.5. □

Example 8.7. Let Q be the UHF-algebra with $K_0(Q) = \mathbb{Q}$. Then by 8.1 Q is not weakly semiprojective with respect to $\mathbf{B}$. In fact, one can easily show, from the proof of this section, that Q is not weakly semiprojective with respect to $\mathcal{O}_\infty$. In fact, any non-elementary matroid C^*-algebra A is not weakly semiprojective with respect to $\mathcal{O}_\infty$, since its $K_0(A)$ is a dense subgroup of $\mathbb{Q}$, which is not free.

On the other hand, if A is a simple AF-algebra with $K_0(A) = \mathbb{Z} + \mathbb{Z}\sqrt{2}$, then A is weakly semiprojective with respect to $\mathbf{B}$ since $K_0(A) \cong \mathbb{Z} \oplus \mathbb{Z}$ as an abelian group.

Acknowledgements

The author acknowledges support from the National Science Foundation and also thanks N. C. Phillips for some helpful conversations. This work was also supported by Shanghai Priority Academic Disciplines from East China Normal University.

References

[BG] R. G. Bartle and L. M. Graves, *Mappings between function spaces*, Trans. Amer. Math. Soc. **72** (1952), 400-413. MR **13:**951i

[B] B. Blackadar, *K-Theory for Operator Algebras*, MSRI Monographs, vol. 5, 2nd edition, Cambridge Press, 1998. MR **99g:**46104

[Br1] L. G. Brown, Stable isomorphisms of hereditary subalgebras of C^*-algebras, Pacific J. Math. **71** (1977), 335-348. MR **56:**12894

[BDF] L. G. Brown, R. Douglas and P. Fillmore, *Extensions of C^*-algebras and K-homology*, Ann. of Math. **105** (1977), 265-324. MR **56:**16399

[CE] M-D. Choi and E. Effros, *The completely positive lifting problem for C^*-algebras*, Ann. of Math. **104** (1976), 585-609. MR **54:**5843

[DE1] M. Dadarlat and S. Eilers, *On the classification of nuclear C^*-algebras*, Proc. London Math. Soc. **85** (2002), 168–210. MR **2003d:**19006

[DE2] M. Dadarlat and S. Eilers, *Asymptotic unitary equivalence in KK-theory*, K-Theory **23** (2001), 305–322. MR **2002h:**19005

[DL1] M. Dadarlat and T. Loring, *K-homology, asymptotic representations and unsuspended E-theory*, J. Funct. Anal. **126** (1994), 367–383. MR **96d:**46092

[DL2] M. Dadarlat and T. Loring, *A universal multi-coefficient theorem for the Kasparov groups*, Duke J. Math. **84** (1996), 355–377. MR **97f:**46109

[DR] K. J. Dykema and M. Rørdam, *Purely infinite simple C^*-algebras arising from free product constructions*, Canad. J. Math. **50** (1998), 323-341. MR **99d:**46074

[Ell1] G. A. Elliott, *On the classification of C^*-algebras of real rank zero*, J. Reine Angew. Math. **443** (1993), 179-219. MR **94i:**46074

[EG] G. A. Elliott and G. Gong, *On the classification of C^*-algebras of real rank zero, II*, Ann. Math. **144** (1996), 497-610. MR **98j:**46055

[F] L. Fuchs, *Infinite Abelian Groups*, Academic Press, New York and London, 1970. MR **41:**333

[GL1] G. Gong and H. Lin, *Almost multiplicative morphisms and almost commuting matrices*, J. Operator Theory **40** (1998), 217–275. MR **2000c:**46105

[GL2] G. Gong and H. Lin, *Almost multiplicative morphisms and K-Theory*, Inter. J. Math. **8** (2000), 983-1000. MR **2001j:**46081

[H] N. Higson, *A characterization of KK-theory*, Pacific J. Math. **126** (1987), 253-276. MR **88a:**46083

[HR] U. Haagerup and M. Rørdam, *Perturbations of the rotation C^*-algebras and of the Heisenberg commutation relation*, Duke Math. J. **77** (1995), 627–656. MR **96e:**46073

[Ka] G. Kasparov, *Hilbert C^*-modules: theorems of Stinespring and Voiculescu*, J. Operator Theory **4** (1980), 133-150. MR **82b:**46074

[K2] E. Kirchberg, *The classification of purely infinite simple C^*-algebras using Kasparov's theory*, to appear in the Fields Institute Communication series.

[KP] E. Kirchberg and N. C. Phillips, *Embedding of exact C^*-algebras in the Cuntz algebra $\mathcal{O}_2$*, J. Reine und Angew. Math. **525** (2000), 17-53. MR **2001d:**46086a

[Ln1] H. Lin, *Almost commuting unitary elements in purely infinite simple C^*-algebras*, Math. Ann. **303** (1995), 599-616. MR **96k:**46101

[Ln2] H. Lin, *Classification of simple C^*-algebras with unique traces*, Amer. J. Math. **119** (1997), 1263-1289. MR **98m:**46088

[Ln3] H. Lin, *Stable approximate unitary equivalences of homomorphisms*, J. Operator Theory **47** (2002), 343–378. MR **2003c:**46082

[Ln4] H. Lin, *Tracially AF C^*-algebras*, Trans. Amer. Math. Soc. **353** (2001), 693–722. MR **2001j:**46089

[Ln5] H. Lin, *Classification of simple TAF C^*-algebras*, Canad. J. Math. **53** (2001), 161–194. MR **2002h:**46102

[Ln6] H. Lin, *Tracial topological rank of C^*-algebras*, Proc. London Math. Soc. **83** (2001), no. 1, 199–234. MR **2002e:**46063

[Ln7] H. Lin, *An introduction to the classification of amenable C^*-algebras*, World Scientific, New Jersey/London/Singapore/Hong Kong/ Bangalore, 2001. MR **2002k:**46141

[Ln10] H. Lin, *An approximate Universal Coefficient Theorem*, preprint.

[Lo1] T. Loring, *C^*-algebras generated by stable relations*, J. Funct. Anal. **112** (1993), 159–203. MR **94k:**46115

[Lo2] T. Loring, *Stable relations. II. Corona semiprojectivity and dimension-drop C^*-algebras*, Pacific J. Math. **172** (1996), no. 2, 461–475. MR **97c:**46070

[Lo3] T. Loring, *Lifting solutions to perturbing problems in C^*-algebras*, Fields Institute Monographs 8, Amer. Math. Soc., Providence, RI, 1997. MR **98a:**46090

[Ped] G. K. Pedersen, *C^*-algebras and their automorphisms*, Academic Press, London, 1979. MR **81e:**46037

[P1] N. C. Phillips, *Approximation by unitaries with finite spectrum in purely infinite C^*-algebras*, J. Funct. Anal. **120** (1994), 98–106. MR **95c:**46092

[P2] N. C. Phillips, *Approximate unitary equivalence of homomorphisms from odd Cuntz algebras*, Fields Inst. Comm. **13** (1997), 243-255. MR **97k:**46068

[P3] N. C. Phillips, *A classification theorem for nuclear purely infinite simple C^*-algebras*, Doc. Math. **5** (2000), 49–114. MR **2001d:**46086b

[Ro1] M. Rørdam, *Classification of inductive limits of Cuntz algebras*, J. Reine Angew. Math. **440** (1993), 175-200. MR **94k:**46120

[Ro2] M. Rørdam, *Classification of certain infinite simple C^*-algebras*, J. Funct. Anal. **131** (1995), 415-458. MR **96e:**46080a

[Ro3] M. Rørdam, *A short proof of Elliott's theorem: $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$*, C. R. Math. Rep. Acad. Sci. Canada **16** (1994), 31-36. MR **95d:**46064

[Ro4] M. Rørdam, *Classification of nuclear, simple C^*-algebras*, in Operator Algebras, VII, Encyclopaedia of Mathematical Sciences, Springer-Verlag, Berlin, Heidelberg and New York, 2002. MR **2003i:**46060

[RS] J. Rosenberg and C. Schochet, *The Kunneth theorem and the universal Coefficient theorem for Kasparov's generalized functor*, Duke Math. J. **55** (1987), 431-474. MR **88i:**46091

[S1] C. Schochet, *Topological methods for C^*-algebras. III, Axiomatic homology*, Pacific J. Math. **114** (1984), 399–445. MR **86g:**46102

[S2] C. Schochet, *Topological methods for C^*-algebras IV, mod p homology*, Pacific J. Math. **114** (1984), 447-468. MR **86g:**46103

[S3] C. L. Schochet, *The fine structure of the Kasparov groups I: Continuity of the KK-paring*, preprint, Sept. 1996.

[S4] C. L. Schochet, *The fine structure of the Kasparov groups II: relative quasidiagonality*, preprint, Sept. 1996.

[Zh1] S. Zhang, *A property of purely infinite simple C^*-algebras*, Proc. Amer. Soc., **109** (1990), 717-720. MR **90k:**46134

[Zh2] S. Zhang, *On the structure of projections and ideals of corona algebras*, Canad. J. Math. **41** (1989), 721–742. MR **90h:**46094

[Zh3] S. Zhang, *On the exponential rank and exponential length of C^*-algebras*, J. Operator Theory **28** (1992), 337–355. MR **95d:**46065

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